ORIENTATIONS IN LEGENDRIAN CONTACT HOMOLOGY AND EXACT LAGRANGIAN IMMERSIONS

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ABSTRACT. We show how to orient moduli spaces of holomorphic disks with boundary on an exact Lagrangian immersion of a spin manifold into complex n-space in a coherent manner. This allows us to lift the coefficients of the contact homology of Legendrian spin submanifolds of standard contact $(2n + 1)$-space from $\mathbb{Z}_2$ to $\mathbb{Z}$. We demonstrate how the $\mathbb{Z}$-lift provides a more refined invariant of Legendrian isotopy. We also apply contact homology to produce lower bounds on double points of certain exact Lagrangian immersions into $\mathbb{C}^n$ and again including orientations strengthens the results. More precisely, we prove that the number of double points of an exact Lagrangian immersion of a closed manifold M whose associated Legendrian embedding has good DGA is at least half of the dimension of the homology of M with coefficients in an arbitrary field if M is spin and in $\mathbb{Z}_2$ otherwise.

CONTENTS

1. INTRODUCTION

Legendrian contact homology has been an effective tool in studying Legendrian submanifolds in $\mathbb{R}^{2n+1}$. In $\mathbb{R}^3$, Chekanov [3] and Eliashberg and Hofer (unpublished but see [7]) used contact homology to show that Legendrian knots are not determined up to Legendrian isotopy by the so-called classical invariants (topological isotopy class, Thurston-Bennequin

number, and Maslov class). Subsequently, contact homology has been used to greatly illuminate the nature of Legendrian knots in $\mathbb{R}^3$. The contact homology of Legendrian submanifolds in $\mathbb{R}^{2n+1} \approx \mathbb{C}^n \times \mathbb{R}$ (for $n > 1$) was given a rigorous foundation in [6] and its efficacy was demonstrated in [5]. Very roughly speaking contact homology is the homology of a differential graded algebra (DGA) associated to a Legendrian submanifold $L \subset \mathbb{C}^n \times \mathbb{R}$. The algebra is generated by double points in the (Lagrangian) projection of L into $\mathbb{C}^n$ and the differential counts rigid holomorphic disk with corners at these double points and boundary on the projected Legendrian submanifold. In the initial definition of contact homology the disks were counted modulo 2 since in that version of the theory orientations and orientability of spaces of holomorphic disks need not be considered. A $\mathbb{Z}$-lift of contact homology of Legendrian knots in $\mathbb{R}^3$ have been introduced in a purely combinatorial fashion in [9]. It is however still not known if the oriented version of the theory in this case is any stronger that the unoriented version of the theory. Orientations for the moduli space of certain Riemann surfaces without boundary has been discussed in [11, 8, 2]. In this paper we show how to lift the DGA of Legendrian submanifolds, of $\mathbb{R}^{2n+1}$, which are spin to $\mathbb{Z}$. We demonstrate that this lift gives a more refined invariant of Legendrian isotopy than does the theory over $\mathbb{Z}_2$ in dimensions $2n + 1 \geq 9$. For Legendrian knots in $\mathbb{R}^3$, our analytical approach to orientations recovers the combinatorial sign rule of [9] and furthermore gives rise to another combinatorial sign rule not mentioned there. We also use Legendrian contact homology to produce lower bounds on the double points of exact Lagrangian immersions into $\mathbb{C}^n$. (A Lagrangian immersion $f : M \to \mathbb{C}^n$ is *exact* if the closed form $f^*(\sum_{j=1}^n y_j \, dx_j)$, where $(x_1 + iy_1, \ldots, x_n + iy_n)$ are standard coordinates on $\mathbb{C}^n$, is exact.) Generically an exact Lagrangian immersion can be lifted to a Legendrian embedding. A DGA is called good if it is (tame) isomorphic to a DGA without constant terms in its differential [3]. We show that if $f : M \to \mathbb{C}^n$ is an exact self-transverse Lagrangian immersion of a closed manifold such that the DGA associated to a Legendrian lift of f is good then the number $R(f)$ of double points of f satisfies

$$(1.1) \qquad\qquad R(f) \geq \frac{1}{2} \dim(H_*(M; \Lambda)),$$

where $\Lambda = \mathbb{Q}$ or $\Lambda = \mathbb{Z}_p$ for any prime p if M is spin and where $\Lambda = \mathbb{Z}_2$ otherwise. It is easy to construct exact Lagrangian immersions of spheres and tori of arbitrary dimensions which shows that the estimate (1.1) is the best possible. While the hypothesis on the exact Lagrangian immersion seems somewhat unnatural it is frequently satisfied and from anecdotal evidence one would expect exact Lagrangian immersions with non-good DGA's to have more double points than ones with good DGA's. Despite this evidence it does not seem straightforward to use contact homology for estimates when the algebra is not good. However, we prove that if one can establish an estimate like (1.1) with any fixed constant subtracted from the right hand side then (1.1) is true too. The paper is organized as follows. In Section 2 we introduce basic notions which will be used throughout the paper. In Section 3 we show how to orient moduli spaces of holomorphic disks relevant to contact homology. To accomplish this we discuss orientations of determinant bundles over spaces of (stabilized) $\bar{\partial}$-operators associated to Legendrian submanifolds and their interplay with orientations of spaces of conformal structures on punctured disks. Similar constructions are carried out in [12] but some of the details differ. In Section 4 we define the DGA associated to a Legendrian spin submanifold L as an algebra over $\mathbb{Z}[H_1(L)]$ with differential ∂ and prove that $\partial^2 = 0$. Furthermore we prove the invariance of contact homology under Legendrian isotopy by a mixture of a homotopy method and the more direct bifurcation analysis, making use of the stabilization mentioned above. (Over $\mathbb{Z}_2$ this invariance proof gives an alternative to the invariance proof given in [6].) We also describe how the contact homology depends on the choice of spin structure of the Legendrian submanifold and we derive diagrammatic sign rules for Legendrian knots

in $\mathbb{R}^3$. In Section 5, we adapt a theorem of Floer [10] to our situation so that, in special cases, the differential in contact homology can be computed. We also apply these results to construct examples which demonstrates that contact homology over $\mathbb{Z}$ is more refined than contact homology over $\mathbb{Z}_2$. In Section 6 we prove the results related to the double point estimate for exact Lagrangian immersion mentioned above.

2. Basic notions

In this section we introduce notation and briefly describe the construction of the contact homology of a Legendrian submanifold in $\mathbb{R}^{2n+1}$. For more details on this construction, see [5, 6]. Let $(x_1, y_1, \ldots, x_n, y_n, z)$ be standard coordinates on $\mathbb{R}^{2n+1}$. Throughout this paper we consider the standard contact structure $\xi = \ker(dz - \sum y_j \, dx_j)$ on $\mathbb{R}^{2n+1}$ and we make use of the following two projections: the *front projection* $\Pi_F : \mathbb{R}^{2n+1} \to \mathbb{R}^{n+1}$ which forgets the y_j-coordinates and the *Lagrangian projection* $\Pi_{\mathbb{C}} : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ which forgets the z-coordinate. Note that if $L \subset \mathbb{R}^{2n+1}$ is a Legendrian submanifold then $\Pi_{\mathbb{C}} : L \to \mathbb{R}^{2n} = \mathbb{C}^n$ is a Lagrangian immersion. The *Maslov class* of a Lagrangian immersion $\phi : M \to \mathbb{C}^n$ is the homomorphism $\mu : H_1(M) \to \mathbb{Z}$ such that $\mu(A)$ is the Maslov index of the loop of Lagrangian planes which consists of tangent planes to $\phi(M)$ along some loop $\gamma \subset M$ representing A. The *Maslov number* $m_\phi > 0$ of ϕ is the positive integer which is a generator of the image of $\mu_\phi : H_1(L; \mathbb{Z}) \to \mathbb{Z}$ if $\mu_\phi \neq 0$ and $m_\phi = 0$ if $\mu_\phi = 0$. If $L \subset \mathbb{C}^n \times \mathbb{R}$ is a Legendrian immersion we sometimes write $m(L)$ for the Maslov number of the Lagrangian immersion $\Pi_{\mathbb{C}} : L \to \mathbb{C}^n$. We define the contact homology of a spin Legendrian submanifold L in $(\mathbb{R}^{2n+1}, \xi)$ equipped with a spin structure as an algebra over the group ring $\mathbb{Z}[H_1(L)]$ if L is connected and as an algebra over $\mathbb{Z}$ otherwise in the following way. Consider first the connected case. Assume that L is generic with respect to the Lagrangian projection and let $\mathcal{A}$ be the free associative (non-commutative) algebra over $\mathbb{Z}[H_1(L)]$ generated by the (transverse) double points of $\Pi_{\mathbb{C}}(L) \subset \mathbb{R}^{2n} = \mathbb{C}^n$. There is a $\mathbb{Z}$ grading on this algebra defined as follows. For a double point c_i denote the two points in L mapping to c_i by c_i^+ and c_i^- where c_i^+ has the larger z-coordinate. Choose, for each double point c_i, a path γ_i in L that runs from c_i^+ to c_i^-. The Conley-Zehnder index $\nu(c_i)$ of c_i is the Maslov index of the loop of Lagrangian subspaces in $\mathbb{C}^n$ which is the path of planes tangent to $\Pi_{\mathbb{C}}(L)$ along γ closed up in a specific way, see [6]. The grading of c_i is defined as $|c_i| = \nu(c_i) - 1$ and the grading of a homology class $A \in H_1(L)$ is defined as the negative of the Maslov index of the loop of Lagrangian subspaces in $\mathbb{C}^n$ which are tangent to $\Pi_{\mathbb{C}}(L)$ along some loop $\gamma \subset L$ representing A. If one defines the algebra over $\mathbb{Z}$ instead of $\mathbb{Z}[H_1(L)]$ the grading is only in $\mathbb{Z}/m(L)\mathbb{Z}$. In the disconnected case $\mathcal{A}$ is graded over $\mathbb{Z}_2$, see Subsection 4.1. (There is also a relative $\mathbb{Z}/m(L)\mathbb{Z}$-grading which we will not discuss in this paper.)

The differential of the algebra $\mathcal{A}$, $\partial : \mathcal{A} \to \mathcal{A}$ lowers grading by 1 and is defined by counting holomorphic disks in $\mathbb{C}^n$ with boundary on $\Pi_{\mathbb{C}}(L)$. More precisely, for a, a double point of $\Pi_{\mathbb{C}}(L)$, $\mathbf{b}$, a word in the double points of $\Pi_{\mathbb{C}}(L)$, and $A \in H_1(L)$, we consider the moduli space $\mathcal{M}_A(a; \mathbf{b})$. This space consists of holomorphic maps from the punctured unit disk to $\mathbb{C}^n$ which maps the boundary to $\Pi_{\mathbb{C}}(L)$, which are asymptotic, see [6], to the double points

specified at the punctures, and which when restricted to the boundary satisfies a certain homology condition specified by A. If L is connected then the differential on $\mathcal{A}$ is

$$(2.1) \qquad \partial a = \sum_{\mathbf{b}} (\#\mathcal{M}_A(a;\mathbf{b}))A\mathbf{b},$$

where the sum runs over all $\mathbf{b}$ such that $\mathcal{M}_A(a;\mathbf{b})$ is 0-dimensional. The moduli spaces which are 0-dimensional are compact manifolds. In Section 3.4 we orient $\mathcal{M}_A(a;\mathbf{b})$. This means in particular that each component of a 0-dimensional moduli space comes equipped with a sign and $\#\mathcal{M}_A(a;\mathbf{b})$ in (2.1) is the algebraic number of points in the moduli space. If L is not connected then the differential is defined as in (2.1) except that the homology class A there should be deleted. The *contact homology* of L is $CH(L) = \mathrm{Ker}(\partial)/\mathrm{Im}(\partial)$. Finally, we note that if $L \subset \mathbb{C}^n \times \mathbb{R}$ is a Legendrian submanifold then double points of $\Pi_{\mathbb{C}} \colon L \to \mathbb{C}^n$ correspond to segments in the $\mathbb{R}$-direction of $\mathbb{C}^n \times \mathbb{R}$ with its endpoints on L. The vector field ∂_z is the Reeb field of the contact form $dz - \sum_j y_j \, dx_j$ and thus such a segment is called a Reeb chord. We will therefore use the words *Reeb chord* and *double point* interchangeably below.

3. Orienting the moduli spaces

In this section we orient moduli spaces of holomorphic disks in $\mathbb{C}^n$ with boundary on $\Pi_{\mathbb{C}}(L)$, where $L \subset \mathbb{C}^n \times \mathbb{R}$ is a Legendrian submanifold which is spin. To orient the moduli spaces we find an orientation of the determinant bundle of a stabilized version of the linearization of the defining $\bar{\partial}$-equation. The source space of the linearization splits into an infinite dimensional space and a finite dimensional space arising from automorphisms or variations of the conformal structure of the source disk. The restriction of the linearization to the infinite dimensional space is a Fredholm operator. We orient the determinant bundles over spaces of such Fredholm operators and orient spaces of automorphisms and conformal structures separately. The orientations we define depend on several choices. We make these choices so that it is possible to endow the graded algebra discussed in Section 2 with a differential. In particular, this differential must respect the multiplication of the algebra in the sense that it satisfies the graded Leibniz rule, see Equation (4.1).

3.1. **Lagrangian boundary conditions for the $\bar{\partial}$-operator on punctured disks.** Let D_m denote the unit disk D in $\mathbb{C}$ with m distinct punctures $\{p_1, \ldots, p_m\}$ on the boundary, $m \geq 0$. The orientation on D induces an orientation on its boundary. If one puncture, p_1 say, is distinguished then the orientation on ∂D_m induces an order of the punctures. The punctures subdivide ∂D_m into m disjoint oriented open arcs. We denote their closures by $I_1, \ldots, I_m$, with notation such that $\partial I_j = p_{j+1} - p_j$ (where $p_0 = p_m$ and $p_{m+1} = p_1$). For convenience we write p_j^+ and p_j^- for p_j thought of as a point in I_{j-1} and I_j, respectively. (We let $I_0 = I_m$, $I_{m+1} = I_1$, and in the special case when $m = 0$, $I_0 = I_1 = \partial D$.) A *Lagrangian boundary condition* on D_m is a collection of maps

$$\lambda = (\lambda_1, \ldots, \lambda_m)$$

where $\lambda_j \colon I_j \to \mathrm{Lag}(n)$, where $\mathrm{Lag}(n)$ denotes the space of Lagrangian subspaces of $\mathbb{C}^n$. For non-punctures $z \in \partial D_m$ we write $\lambda(z)$ to denote $\lambda_j(z)$ where j is the unique subscript such that $z \in I_j$. If p is a puncture then $p \in I_{j-1}$ and $p \in I_j$. We write $\lambda(p^+) = \lambda_{j-1}(p^+)$ and $\lambda(p^-) = \lambda_j(p_j^-)$. An *oriented Lagrangian boundary condition* on D_m is a collection of maps

$$\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_m)$$

where $\tilde{\lambda}_j \colon I_j \to \mathrm{Lag}_0(n)$, where $\mathrm{Lag}_0(n)$ denotes the space of oriented Lagrangian subspaces of $\mathbb{C}^n$. Note that any Lagrangian boundary condition on a disk with at least one puncture lifts (non-uniquely) to an oriented boundary condition, and that there are boundary conditions

on the 0-punctured disk which do not lift. A *trivialized Lagrangian boundary condition* for the $\bar{\partial}$-operator on D_m is a collection of maps

$$A = (A_1, \ldots, A_m),$$

where $A_j \colon I_j \to U(n)$. A trivialized Lagrangian boundary condition induces a Lagrangian boundary condition λ via $\lambda_j(z) = A_j(z)\mathbb{R}^n$ and a trivialization $(v_1(z), \ldots, v_n(z))$ of a Lagrangian boundary condition (i.e. an ON-basis in λ) gives a trivialized Lagrangian boundary condition by defining $A(z)$ as the matrix with column vectors $(v_1(z), \ldots, v_n(z))$. Fixing an orientation on $\mathbb{R}^n$, a trivialized boundary condition can be considered as an oriented boundary condition.

3.1.1. *Angles, weights, and Fredholm properties.* Neighborhoods of the punctures of D_m will be thought of as infinite half strips $[0, \infty) \times [0, 1]$ or $(-\infty, 0] \times [0, 1]$ with coordinates $\tau + it$, and we consider the $\bar{\partial}$-operator

(3.1) $$\bar{\partial} \colon \mathcal{H}_{2,\epsilon}[\lambda](D_m, \mathbb{C}^n) \to \mathcal{H}_{1,\epsilon}[0](D_m, T^{*0,1}D_m \otimes \mathbb{C}^n),$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_m) \in \mathbb{R}^m$, and $\mathcal{H}_{2,\epsilon}[\lambda]$ is the closed subspace of the Sobolev space $\mathcal{H}_{2,\epsilon}$, with weight $e^{\epsilon_j|\tau|}$ in the neighborhood of the jth puncture, which consists of elements u with $u(z) \in \lambda_j(z)$ for $z \in I_j$, and $\bar{\partial}u = 0$ on ∂D_m and where $\mathcal{H}_{1,\epsilon}[0]$ is the closed subspace of the Sobolev space $\mathcal{H}_{1,\epsilon}$ which consist of elements with vanishing trace (restriction to the boundary). See [6], Section 5. We denote the operator in (3.1) by $\bar{\partial}_{\lambda,\epsilon}$. Often in our applications, the weight will be clear from the context and we drop it from the notation writing simply $\bar{\partial}_\lambda$.

We recall the following notion from [6]. Let V_1 and V_2 be Lagrangian subspaces of $\mathbb{C}^n$. Define the *complex angle* $\theta(V_1, V_2) \in [0, \pi)^n$ inductively as follows. If $\dim(V_1 \cap V_2) = r \geq 0$ let $\theta_1 = \cdots = \theta_r = 0$ and let $\mathbb{C}^{n-r}$ denote the Hermitian complement of $\mathbb{C} \otimes (V_1 \cap V_2)$ and let $V_j' = V_j \cap \mathbb{C}^{n-r}$ for $j = 1, 2$. If $\dim(V_1 \cap V_2) = 0$ then let $V_j' = V_j$, $j = 1, 2$ and let $r = 0$. Then V_1' and V_2' are Lagrangian subspaces. Let α be smallest angle such that $\dim(e^{i\alpha}V_1' \cap V_2') = r' > 0$. Let $\theta_{r+1} = \cdots = \theta_{r+r'} = \alpha$. Repeat the construction until θ_n has been defined. Let $\lambda \colon \partial D_m \to \text{Lag}(n)$ be a Lagrangian boundary condition. Let $\theta^j = (\theta_1^j, \ldots, \theta_n^j) \in [0, \pi)^n$ denote the complex angle of $\lambda(p_j^+)$ and $\lambda(p_j^-)$. As shown in [5], the operator $\bar{\partial}_{\lambda,\epsilon}$ is Fredholm when

(3.2) $$\epsilon_j \neq -\theta_k^j + l\pi, \text{ for all } 1 \leq j \leq m, \, 1 \leq k \leq n, \text{ and all } l \in \mathbb{Z}.$$

The *determinant line* $\det(F)$ of a Fredholm operator $F \colon B_1 \to B_2$ between Banach spaces is defined as the tensor product of the highest exterior power of its kernel and the highest exterior power of the dual of its cokernel, respectively. That is,

$$\det(F) = \Lambda^{\max} \text{Ker}(F) \otimes \Lambda^{\max} \text{Coker}(F)^*.$$

Let

$$X_m \subset C^\infty \left(\sqcup_{j=1}^m I_j; \, \text{Lag}(n) \right) \times \mathbb{R}^m$$

denote the space of all Lagrangian boundary conditions and weights for which (3.2) is satisfied. It is a standard result that there exists a determinant line bundle E over X_m with fiber over (λ, ϵ) equal to $\det(\bar{\partial}_{\lambda,\epsilon})$. The bundle E is not orientable. In fact, the corresponding determinant bundle over oriented Lagrangian boundary conditions on the 0-punctured disk is non-orientable. What is needed for orientability in that case are trivialized boundary conditions, see Lemma 3.8.

3.1.2. *Parity and signs of punctures.* We say a boundary condition $\lambda\colon \partial D_m \to \mathrm{Lag}(n)$, $m > 0$ is *transverse* if $\lambda(p^+)$ and $\lambda(p^-)$ are transverse for each puncture p. We will subdivide the punctures on a disk with such boundary conditions into four classes. The first subdivision seems rather ad hoc at this point but in our applications to Legendrian submanifolds it has a clear geometric meaning: each puncture has a sign. That is, there are *positive* and *negative* punctures. (In the applications, disks have one positive puncture and all other negative.) The second subdivision comes from the *parity* of a puncture defined in the following way. Let V_1 and V_2 be two *oriented* transverse Lagrangian subspaces of $\mathbb{C}^n$. Let $\theta = \theta(V_1, V_2)$ be the complex angle of (V_1, V_2). Then there exists complex coordinates $(z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n)$ on $\mathbb{C}^n$ so that in these coordinates

$$V_1 = \mathrm{Span}(\partial_{x_1}, \ldots, \partial_{x_n})$$

and

$$V_2 = \mathrm{Span}(e^{i\theta_1}\partial_{x_1}, \ldots, e^{i\theta_n}\partial_{x_n}).$$

We will call such coordinates *canonical coordinates of* (V_1, V_2). Note that the canonical coordinates are not unique. However, in the constructions below the choice of specific canonical coordinates will be irrelevant. For example, the unitary linear map of $\mathbb{C}^n$ with matrix

$$(3.3) \qquad\qquad \mathrm{Diag}\left(e^{-i(\pi-\theta_1)}, e^{-i(\pi-\theta_2)}, \ldots, e^{-i(\pi-\theta_n)}\right),$$

in canonical coordinates is well-defined and maps V_1 isomorphically to V_2. The ordered pair (V_1, V_2) is *even (odd)* if (3.3) is an orientation reversing (preserving) map from V_1 to V_2. Consider a transverse oriented boundary condition $\lambda\colon \partial D_m \to \mathrm{Lag}_0(n)$. If p is a negative puncture then p is *even* if the pair $(\lambda(p^+), \lambda(p^-))$ is even and p is *odd* if $(\lambda(p^+), \lambda(p^-))$ is odd. If q is a positive puncture then q is *even* (*odd*) if the pair $(\lambda(q^-), \lambda(q^+))$ is even (odd).

3.2. **Gluing operations.** We define gluing operations for the boundary conditions discussed in Subsection 3.1 and explain how orientations of the determinant lines over the pieces relate to an orientation of the determinant line of the resulting glued boundary condition. More precisely, the gluing operations give rise to exact sequences involving kernels and cokernels of operators and the orientations of the determinant lines are related via these. In our applications, the exact form of the relations is important. We therefore start out with explaining our conventions for exact sequences of oriented vector spaces.

3.2.1. *Orientation conventions.* All our gluing operations give rise to exact sequences with at most four non-zero terms so we discuss only this case. Let

$$(3.4) \qquad\qquad 0 \longrightarrow V_1 \xrightarrow{\ \alpha\ } W_1 \xrightarrow{\ \beta\ } W_2 \xrightarrow{\ \gamma\ } V_2 \longrightarrow 0,$$

be an exact sequence of finite dimensional vector spaces. This sequence induces an isomorphism

$$(3.5) \qquad\qquad \Lambda^{\mathrm{max}}V_1 \otimes \Lambda^{\mathrm{max}}V_2^* \approx \Lambda^{\mathrm{max}}W_1 \otimes \Lambda^{\mathrm{max}}W_2^*.$$

Our interest in this isomorphism is its effect on orientations. Note that there is a natural correspondence between orientations on a finite dimensional vector space V and on its dual V^*. To see this, think of an orientation of V as a non-zero element in $\Lambda^{\mathrm{max}}V$ up to multiplication by a positive number. The correspondence can then be obtained as follows. Pick any basis $v_1, \ldots, v_n$ in V and let $v_1^*, \ldots, v_n^*$ be the dual basis in V^*. Now identify the orientation in V given by

$$v_1 \wedge \cdots \wedge v_n$$

with the orientation in V^* given by

$$v_1^* \wedge \cdots \wedge v_n^*.$$

It is easy to see that this identification is independent of the choice of basis. If V and W are finite dimensional vector spaces then the correspondence just described gives rise to a correspondence between orientations of the 1-dimensional vector space

$$\Lambda^{\max}V \otimes \Lambda^{\max}W^*$$

and pairs (o_V, o_W) of orientations of V and W respectively, modulo the following equivalence relation: a pair represented by $(\phi_V, \phi_W) \in \Lambda^{\max}V \times \Lambda^{\max}W$ is identified with the pair represented by $(a\phi_V, a\phi_W)$ for any non-zero $a \in \mathbb{R}$. We call an equivalence class an *orientation pair*. Using this terminology, we interpret the isomorphism (3.5) on the orientation level as follows: the sequence (3.4) gives a correspondence between orientation pairs of (V_1, V_2) with orientation pairs of (W_1, W_2). We next give a concrete description of this correspondence. Let ξ be an orientation pair on (V_1, V_2). Pick bases $v_1^1, \ldots, v_1^m$ in V_1 and $v_2^1, \ldots, v_2^s$ in V_2 such that

$$\left(v_1^1 \wedge \cdots \wedge v_1^m, v_2^1 \wedge \cdots \wedge v_2^s\right),$$

represents ξ. Pick vectors $w_1^{m+1}, \ldots, w_1^n$ such that $\alpha(v_1), \ldots, \alpha(v_m), w_1^{m+1}, \ldots, w_1^n$ is a basis of W_1. By exactness, the vectors $\beta(w_1^{m+1}), \ldots, \beta(w_1^n)$ are linearly independent in W_2. Let $w_2^1, \ldots, w_2^s$ be any vectors in W_2 such that $\gamma(w_2^j) = v_2^j$ for all j. Define $\phi(\xi)$ to be the orientation pair represented by

$$\left(\alpha(v_1^1) \wedge \cdots \wedge \alpha(v_1^m) \wedge w_1^{m+1} \wedge \cdots \wedge w_1^n, \beta(w_1^n) \wedge \cdots \wedge \beta(w_1^{m+1}) \wedge w_2^1 \wedge \cdots \wedge w_2^s\right).$$

(Note the *reverse order* of the vectors $\beta(w_1^j)$.) It is easy to see that $\phi(\xi)$ is well defined. The inverse ψ of ϕ can be described as follows. Let η be any orientation pair of (W_1, W_2). Pick bases $w_1^1, \ldots, w_1^n$ in W_1 and $w_2^1, \ldots w_2^t$ in W_2 such that

$$\left(w_1^1 \wedge \cdots \wedge w_1^n, w_2^1 \wedge \cdots \wedge w_2^t\right)$$

represents η. Pick any basis $v_1^1, \ldots, v_1^m$ in V_1 then there are vectors $\hat{w}_1^{m+1}, \ldots, \hat{w}_1^n$ in W_1 so that

$$w_1^1 \wedge \cdots \wedge w_1^n \quad \text{and} \quad \alpha(v_1^1) \wedge \cdots \wedge \alpha(v_1^m) \wedge \hat{w}_1^{m+1} \wedge \cdots \wedge \hat{w}_1^n$$

represent the same orientation on W_1. Furthermore, there are vectors $\hat{w}_2^1, \ldots, \hat{w}_2^s$ such that

$$w_2^1 \wedge \cdots \wedge w_2^t \quad \text{and} \quad \beta(\hat{w}_1^n) \wedge \cdots \wedge \beta(\hat{w}_1^{m+1}) \wedge \hat{w}_2^1 \wedge \cdots \wedge \hat{w}_2^s$$

represent the same orientation on W_2. Define $\psi(\eta)$ to be the orientation pair on (V_1, V_2) represented by

$$\left(v_1^1 \wedge \cdots \wedge v_1^m, \gamma(\hat{w}_2^1) \wedge \cdots \wedge \gamma(\hat{w}_2^s)\right).$$

It is easy to verify that ψ is well defined and that ψ and ϕ are inverses of each other.

3.2.2. *Two gluing operations.* Let $\lambda \colon \partial D_m \to \operatorname{Lag}(n)$ be a transverse boundary condition on the punctured disk. Let $\bar{\partial}_\lambda$ be the operator with this boundary condition and with trivial weights. Then, by Proposition 5.14 in [6],

$$\operatorname{Index}(\bar{\partial}_\lambda) = n + \mu(\hat{\lambda}),$$

where μ is the Maslov index and where $\hat{\lambda}$ is the loop in $\operatorname{Lag}(n)$ obtained from the map $\lambda \colon \partial D_m \to \operatorname{Lag}(n)$ as follows. Let $\theta \in (0, \pi)^n$ denote the complex angle between $\lambda(p_j^+)$ and $\lambda(p_j^-)$ then there exists coordinates $(x_1 + iy_1, \ldots, x_n + iy_n)$ in $\mathbb{C}^n$ such that

$$\lambda(p_j^+) = \operatorname{Span}(\partial_{x_1}, \ldots, \partial_{x_n})$$

and

$$\lambda(p_j^-) = \operatorname{Span}(e^{i\theta_1}\partial_{x_1}, \ldots, e^{i\theta_n}\partial_{x_n}).$$

Let $\gamma_j(s)$, $0 \leq s \leq 1$, be the path of Lagrangian subspaces given in these coordinates as

$$\text{Span}(e^{-i(\pi-\theta_1)s}\partial_{x_1}, \ldots, e^{-i(\pi-\theta_n)s}\partial_{x_n}).$$

Then $\hat{\lambda}$ is the loop obtained by concatenation of the paths in λ with the paths γ_j. That is,

$$\hat{\lambda} = \lambda_1 * \gamma_2 * \lambda_2 * \cdots * \lambda_m * \gamma_1.$$

Let $A\colon \partial D_m \to U(n)$ and $B\colon \partial D_s \to U(n)$ be trivialized Lagrangian boundary conditions. Let $p \in D_m$ and $q \in D_s$ be points which are not punctures and assume that $A(p) = B(q)$. We assume also that A and B are constant in neighborhoods of p and q, respectively. (If this is not the case, we may homotope the boundary conditions so that they become constant around p and q. Note that such homotopies may change the dimensions of the kernels and cokernels of the operators involved. However, if the homotopy can be chosen sufficiently C^1-small then the dimensions can be kept constant.) We glue D_m and D_s to one disk D_{m+s} with a trivialized Lagrangian boundary condition which is the concatenation of A and B. To define this operation more accurately we proceed as follows. Puncture the disks D_m at p and D_s at q. Use conformal coordinates $[0, \infty) \times [0, 1]$ in a neighborhood of p in D_m and conformal coordinates $(-\infty, 0] \times [0, 1]$ in a neighborhood of q in D_s. Since the boundary conditions A and B are constant in neighborhoods of p and q, respectively, it follows that for all sufficiently large $\rho > 0$ the boundary condition A is constant in $[\rho, \infty) \times (\partial[0, 1])$ and the boundary condition B is constant in $(-\infty, -\rho] \times (\partial[0, 1])$. For such ρ we define a new disk $D(\rho)$ with $m + s$ punctures as follows. Let $D'(\rho) = D_m \setminus ([\rho, \infty) \times [0, 1])$, let $D''(\rho) = D_s \setminus ((-\infty, -\rho] \times [0, 1])$, and let $D(\rho)$ be the disk which is obtained from gluing $D'(\rho)$ to $D''(\rho)$ by identifying $\{\rho\} \times [0, 1]$ with $\{-\rho\} \times [0, 1]$. Note that the boundary conditions glue in a natural way to a boundary condition $A\sharp B$ on $D(\rho)$. Since the Maslov index is additive under this gluing we find that

$$(3.6) \qquad \text{Index}(\bar{\partial}_{A\sharp_\rho B}) = n + \mu(\hat{A}) + \mu(\hat{B}) = \text{Index}(\bar{\partial}_A) + \text{Index}(\bar{\partial}_B) - n.$$

We call this type of gluing *gluing at a boundary point*.

Lemma 3.1. *For all sufficiently large ρ there exists an exact sequence*

$$(3.7) \qquad 0 \longrightarrow \text{Ker}(\bar{\partial}_{A\sharp_\rho B}) \xrightarrow{\alpha_\rho} \text{Ker}(\bar{\partial}_A) \oplus \text{Ker}(\bar{\partial}_B) \xrightarrow{\beta_\rho}$$
$$\text{Coker}(\bar{\partial}_A) \oplus \mathbb{R}^n \oplus \text{Coker}(\bar{\partial}_B) \xrightarrow{\gamma_\rho} \text{Coker}(\bar{\partial}_{A\sharp_\rho B}) \longrightarrow 0$$

where $\mathbb{R}^n$ is naturally identified with $A(p)\mathbb{R}^n = B(q)\mathbb{R}^n$. (We describe the maps α_ρ, β_ρ, and γ_ρ and the identification in Remark 3.3 below.) In particular, together with an orientation on $\mathbb{R}^n$ the above sequence induces an isomorphism between

$$\Lambda^{\max}(\text{Ker}(\bar{\partial}_A) \oplus \text{Ker}(\bar{\partial}_B)) \otimes \Lambda^{\max}(\text{Coker}(\bar{\partial}_A) \oplus \text{Coker}(\bar{\partial}_B))^*$$

and

$$\Lambda^{\max} \text{Ker}(\bar{\partial}_{A\sharp B}) \otimes \Lambda^{\max} \text{Coker}(\bar{\partial}_{A\sharp B})^*.$$

Remark 3.2. The special case $m = s = 0$ of this lemma (together with Lemmas 3.8 and 3.11 below) fixes a misstatement in Lemma 23.5 of [12] (on page 206). Consider two bundles of index 0 with no kernel and no cokernel then Lemma 23.5 implies that the kernel and cokernel of the glued problem are trivial as well. However this cannot be correct since the problem has index $-n$ by (3.6).

Proof. Using Sobolev spaces with a small negative exponential weight $-\delta$, $\delta > 0$ at the punctures $p \in D_m$ and $q \in D_s$ (i.e. in coordinates $\tau + it \in \mathbb{R} \times [0, 1]$ the weight functions w' and w'' equal $e^{-\delta|\tau|}$ in neighborhoods of the punctures and equal 1 elsewhere), we get a

canonical identification of the kernels and cokernels on the punctured and the non-punctured disks, see Lemma 5.2 in [6]. Consider the operator

$$\bar\partial\colon \mathcal{H}_{2,\rho}[A\sharp B](D(\rho),\mathbb{C}^n)\to, \mathcal{H}_{1,\rho}[0](D(\rho),T^{*0,1}D(\rho)\otimes\mathbb{C}^n),$$

where the subscript ρ indicates that we use Sobolev norms with weight functions which are the gluing of the weight functions on the two disks. (Since the support of the glued weight function in $D(\rho)$ is compact this weight function is not very important but it will be convenient to use that norm in the estimates below.) More precisely, this weight function is a smoothing of the function which equals w' on $D_m \setminus ([\rho,\infty)\times[0,1])$ and w'' on $D_s\setminus((-\infty,-\rho]\times[0,1])$. Pick bases $a_1,\ldots,a_{r_1}$ and $b_1,\ldots,b_{r_2}$ in $\mathrm{Ker}(\bar\partial_A)$ and $\mathrm{Ker}(\bar\partial_B)$, respectively. Also, pick bases $\alpha_1,\ldots,\alpha_{s_1}$ and $\beta_1,\ldots,\beta_{s_2}$ in $\mathrm{Coker}(\bar\partial_A)$ and $\mathrm{Coker}(\bar\partial_B)$, respectively. (A priori, elements in the cokernels are elements in the dual of $\mathcal{H}_1[0]$. However, they can be represented by L^2-pairing with smooth functions, see Lemma 5.1 in [6], and we will think of them as such.) Let ϕ'_ρ be a cut-off function which equals 1 on $D_m \setminus ([\tfrac12\rho,\infty)\times[0,1])$, which equals 0 on $D(\rho)\setminus(D_m\setminus[\rho,\infty)\times[0,1])$ and with $|d^k\phi|=\mathcal{O}(\rho^{-1})$ for $k=1,2$. Let ϕ''_ρ be a similar cut-off function on with support in the part of $D(\rho)$ corresponding to D_s. Define $V_\rho\subset\mathcal{H}_{2,\rho}(D(\rho),\mathbb{C}^n)$ to be the L^2-complement of the r_1+r_2 functions

$$\phi'_\rho a_1,\ldots,\phi'_\rho a_{r_1},\phi''_\rho b_1,\ldots,\phi''_\rho b_{r_2}.$$

We claim that there exists a constant C such that for all sufficiently large ρ

$$\|v\|_{2,\rho}\le C\|\bar\partial v\|_{1,\rho},\ \text{for } v\in V_\rho.$$

Assume not, then there exists a sequence of functions $v_j\in V_{\rho(j)}$, $\rho(j)\to\infty$ as $j\to\infty$ such that

(3.8) $$\|v_\rho(j)\|_{2,\rho(j)}=1$$

(3.9) $$\|\bar\partial v_{\rho(j)}\|_{1,\rho(j)}\to 0,\ \text{as } j\to\infty.$$

Let $\Theta_{a,\rho}\subset D(\rho)$ denote the subset

$$[a,\rho]\times[0,1]\cup[-\rho,-a]\times[0,1],$$

where the first part is a subset of D_m and the second of D_s. Let ψ_ρ be a cut-off which equals 1 in $\Theta_{\frac14\rho,\rho}$, which equals 0 in the complement of $\Theta_{0,\rho}$ and which has $|d^k\psi|=\mathcal{O}(\rho^{-1})$, $k=1,2$. Then using the elliptic estimate for the $\bar\partial$-operator on $\mathbb{R}\times[0,1]$ with small positive exponential weight $e^{\epsilon|\tau|}$ at both ends we find using (3.9) that

(3.10) $$\|\psi_{\rho(j)}v_{\rho(j)}\|_{2,\rho(j)}\to 0,\ \text{as } j\to\infty.$$

Let ψ'_ρ be a cut off function which is 1 on $D_m\setminus([\tfrac14\rho,\infty)\times[0,1])$, which is 0 outside $D_m\setminus([\rho,\infty)\times[0,1])$, and with $|d^k\psi'_\rho|=\mathcal{O}(\rho^{-1})$. Using the elliptic estimate on the L^2-complement of the kernel of $\bar\partial$ on D_m we find

(3.11) $$\|\psi'_{\rho(j)}v_{\rho(j)}\|_{2,\rho(j)}\to 0,\ \text{as } j\to\infty.$$

(Note since the support of the cut-off functions ϕ'_ρ are contained in the support of the cut-off functions ψ'_ρ we know $\psi'_{\rho(j)}v_{\rho(j)}$ is orthogonal to the kernel of $\bar\partial$ on D_m.) With ψ''_ρ similarly defined but with support on D_s instead we find

(3.12) $$\|\psi''_{\rho(j)}v_{\rho(j)}\|_{2,\rho(j)}\to 0,\ \text{as } j\to\infty.$$

However, (3.10), (3.11), and (3.12) contradicts (3.8) and we find the estimate on V_ρ holds as claimed. The restriction of $\bar\partial$ to V_ρ is a Fredholm operator of index

$$-\dim(\mathrm{Coker}(\bar\partial_A))-\dim(\mathrm{Coker}(\bar\partial_B))-n.$$

Consider the elements $\phi'\alpha_1, \ldots, \phi'\alpha_{s_1}$ and $\phi''\beta_1, \ldots, \phi''\beta_{s_2}$ in the dual of $\mathcal{H}_{1,\rho}[0]$. Furthermore, we pick in this dual constant functions $c_1, \ldots, c_n$ on the gluing region with values in the orthogonal complement of $A(1)\mathbb{R}^n$ and consider $\psi_\rho c_1, \ldots, \psi_\rho c_n$ as elements in the dual. Moreover, choose them so that their L^2-norms (in the dual weight, which is large in the gluing region) are 1. Now pick elements (in $\mathcal{H}_{1,\rho}[0]$) dual to these elements and all of norm one. Then they stay a uniform positive distance away from the intersection

$$W = \bigcap_j \mathrm{Ker}(\psi'_\rho \alpha_j) \cap \bigcap_k \mathrm{Ker}(\psi''_\rho \beta_k) \cap \bigcap_l \mathrm{Ker}(\psi_\rho c_l),$$

and thus give a direct sum decomposition

$$\mathcal{H}_{1,\rho}[0] = W \oplus \mathbb{R}^{s_1+s_2+n}.$$

Let π be the projection induced by the above direct sum decomposition. We claim that

$$\pi \circ \bar{\partial} \colon V_\rho \to W,$$

is an isomorphism. Assume not, then there is a sequence u_ρ with

$$\|u_\rho\|_{V_\rho} = 1 \text{ and } \|\bar{\partial} u_\rho\|_W \to 0.$$

We need only note that also $\xi(\bar{\partial} u_\rho) \to 0$ for ξ one of the chosen elements in the dual to see that this contradicts the previous estimate. But this is clear since $\partial \hat{\xi} = 0$ where $\hat{\xi}$ is the function which was cut-off to get ξ.

We thus find the exact sequence

$$0 \longrightarrow \mathrm{Ker}(\bar{\partial}_{A\sharp_\rho B}) \xrightarrow{\alpha_\rho} \mathrm{Ker}(\bar{\partial}_A) \oplus \mathrm{Ker}(\bar{\partial}_B) \xrightarrow{\beta_\rho}$$

$$\mathrm{Coker}(\bar{\partial}_A) \oplus \mathbb{R}^n \oplus \mathrm{Coker}(\bar{\partial}_B) \xrightarrow{\gamma_\rho} \mathrm{Coker}(\bar{\partial}_{A\sharp_\rho B}) \longrightarrow 0$$

where α_ρ is the inclusion followed by orthogonal projection to the subspace spanned by $\{\psi'_\rho a_j\}_{j=1}^{r_1} \cup \{\psi''_\rho b_k\}_{k=1}^{r_2}$, where β_ρ is the $\bar{\partial}$-operator followed by second projection in the direct sum decomposition of $\mathcal{H}_{1,\rho}[0]$, and where γ_ρ is the inclusion followed by the projection to the quotient $\mathcal{H}_{1,\rho}[0]/\mathrm{Im}(\bar{\partial})$. $\qquad\square$

Remark 3.3. We note that the natural identification of $\mathbb{R}^n$ in the above sequence is obtained by identifying the cokernel of the $\bar{\partial}$-problem on the gluing strip with the constant functions taking values in $i \cdot A(1)\mathbb{R}^n$. The map α_ρ can be described as follows. First we map the space $(\mathrm{Ker}(\bar{\partial}_A) \oplus \mathrm{Ker}(\bar{\partial}_B))$ into $\mathcal{H}_2(D_\rho, \mathbb{C}^n)$ by cutting off kernel elements with cut-off functions as in the proof. The obtained map is clearly injective. The map α_ρ is then simply the inclusion of $\mathrm{Ker}(\bar{\partial}_{A\sharp B})$ into $\mathcal{H}_2(D_\rho, \mathbb{C}^n)$ followed by the L^2-orthogonal to the image of $(\mathrm{Ker}(\bar{\partial}_A) \oplus \mathrm{Ker}(\bar{\partial}_B))$. The map β_ρ can be described as follows. First we map the space $\mathrm{Coker}(\bar{\partial}_A) \oplus \mathbb{R}^n \oplus \mathrm{Coker}(\bar{\partial}_B)$ into the space $\mathcal{H}_1(D_\rho, T^{*0,1}D_\rho \otimes \mathbb{C}^n)$ using cut-off functions as in the proof above. The map is injective. We then consider $(\mathrm{Ker}(\bar{\partial}_A) \oplus \mathrm{Ker}(\bar{\partial}_B))$ as a subspace of $\mathcal{H}_2(D_\rho, \mathbb{C}^n)$ and map it to the subspace of $\mathcal{H}_1(D_\rho, T^{*0,1}D_\rho \otimes \mathbb{C}^n)$ with the $\bar{\partial}$-operator followed by L^2-orthogonal projection. The map γ_ρ is simply mapping $\mathrm{Coker}(\bar{\partial}_A) \oplus \mathbb{R}^n \oplus \mathrm{Coker}(\bar{\partial}_B)$, included into $\mathcal{H}_1(D_\rho, T^{*0,1}D_\rho \otimes \mathbb{C}^n)$ as above, to $\mathrm{Coker}(\bar{\partial}_{A\sharp B})$ by projection to the quotient.

We next consider another type of gluing. Let $A \colon \partial D_m \to U(n)$ and $B \colon \partial D_s \to U(n)$ be as above. Let p be a puncture of D_m and let q be a puncture on D_r. Assume that $A(p^+) = B(q^-)$ and that $A(p^-) = B(q^+)$ and that A and B are constant in neighborhoods of $p^\pm$ and $q^\pm$, respectively. (In general, our assumption that boundary conditions are C^2-small in standard coordinates around the punctures implies that we can actually homotope them to constant without changing the kernel/cokernel dimension.) Choose conformal coordinates $[0,\infty) \times [0,1]$ in a neighborhood of p and conformal coordinates $(-\infty, 0] \times [0,1]$ in a neighborhood of q. For $\rho > 0$ large enough we can glue $D_m \setminus ([\rho, \infty) \times [0,1])$ to $D_s \setminus ((-\infty, -\rho] \times [0,1])$ to obtain

a disk $D(\rho)$ with $m + r - 2$ punctures. Also the boundary conditions glue in a natural way to a boundary condition $A \sharp B$ on $D(\rho)$. The index of the corresponding operator satisfies

$$\text{Index}(\bar{\partial}_{A \sharp B}) = n + \mu(\widehat{A \sharp B}) = \text{Index}(\bar{\partial}_A) + \text{Index}(\bar{\partial}_B).$$

(Here the equality follows from the fact that to obtain $\widehat{A \sharp B}$ from $\hat{A}$ and $\hat{B}$, n negative π-rotations are removed at their common puncture.) We call this type of gluing, *gluing at a puncture*.

Lemma 3.4. *For all sufficiently large $\rho > 0$ there exist an exact sequence*

$$
(3.13) \qquad
\begin{array}{c}
0 \longrightarrow \text{Ker}(\bar{\partial}_{A \sharp B}) \xrightarrow{\alpha_\rho} \text{Ker}(\bar{\partial}_A) \oplus \text{Ker}(\bar{\partial}_B) \xrightarrow{\beta_\rho} \\[4pt]
\text{Coker}(\bar{\partial}_A) \oplus \text{Coker}(\bar{\partial}_B) \xrightarrow{\gamma_\rho} \text{Coker}(\bar{\partial}_{A \sharp B}) \longrightarrow 0
\end{array}
$$

where α_ρ, β_ρ, and γ_ρ are defined similarly to the maps in Remark 3.3. In particular, orientations of the determinants spaces

$$\Lambda^{\max}\Big(\text{Ker}(\bar{\partial}_A) \oplus \text{Ker}(\bar{\partial}_B)\Big) \otimes \Lambda^{\max}\Big(\text{Coker}(\bar{\partial}_A) \oplus \text{Coker}(\bar{\partial}_B)\Big)^*$$

induces, via (3.13), an orientation on

$$\Lambda^{\max}\text{Ker}(\bar{\partial}_{A \sharp B}) \otimes \Lambda^{\max}\text{Coker}(\bar{\partial}_{A \sharp B})^*.$$

Proof. The proof of this result is similar to the proof of Lemma 3.1. However, in the present situation the proof is simpler since the operator in the gluing region is Fredholm of index 0 with trivial kernel and cokernel. This is also the reason for the absence of the $\mathbb{R}^n$ summand in the third term of the gluing sequence. $\qquad\square$

3.2.3. *Associativity of orientations under gluing.* We show that orientations behave associatively with respect to both gluings at punctures and at boundary points. We will often deal with many vector spaces. For convenience we employ the following notation: if $(V_1, V_2, \ldots, V_n)$ are ordered vector spaces then we denote their direct sum by a column vector containing the vector spaces. That is,

$$V_1 \oplus \cdots \oplus V_n = \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix}.$$

(Note that, when dealing with *oriented* vector spaces the ordering of the terms in a direct sum is important.) We first consider the case of two gluings at punctures. Let $A \colon \partial D_m \to U(n)$, $B \colon \partial D_r \to U(n)$, and $C \colon \partial D_s \to U(n)$ be trivialized transverse boundary conditions for the $\bar{\partial}$-operator on three punctured disks. Assume that A can be glued to B at a puncture and that B can be glued to C at another puncture. Performing these two gluings we obtain a boundary condition $E \colon \partial D_{m+r+s-4} \to U(n)$. Fix orientations o_A, o_B, and o_C on $\det(\bar{\partial}_A)$, $\det(\bar{\partial}_B)$, and $\det(\bar{\partial}_C)$, respectively. Via Lemma 3.4 these orientations induce orientations $o_{A \sharp B}$ and $o_{B \sharp C}$ on $\det(\bar{\partial}_{A \sharp B})$ and $\det(\bar{\partial}_{B \sharp C})$.

Lemma 3.5. *Let o_E' be the orientation on $\det(\bar{\partial}_E)$ induced via gluing (as in Lemma 3.4) from the orientations $o_{A \sharp B}$ and o_C on $\det(\bar{\partial}_{A \sharp B})$ and $\det(\bar{\partial}_C)$, respectively. Let o_E'' be the orientation on $\det(\bar{\partial}_E)$ induced via gluing (as in Lemma 3.4) from the orientations of o_A and $o_{B \sharp C}$ on $\det(\bar{\partial}_A)$ and $\det(\bar{\partial}_{B \sharp C})$, respectively. Then*

$$o_E' = o_E'' = o_E,$$

where o_E is the orientation induced from the exact sequence

$$0 \longrightarrow \operatorname{Ker}(\bar{\partial}_E) \xrightarrow{\alpha_\rho} \begin{bmatrix} \operatorname{Ker}(\bar{\partial}_A) \\ \operatorname{Ker}(\bar{\partial}_B) \\ \operatorname{Ker}(\bar{\partial}_C) \end{bmatrix} \xrightarrow{\beta_\rho}$$

$$\begin{bmatrix} \operatorname{Coker}(\bar{\partial}_A) \\ \operatorname{Coker}(\bar{\partial}_B) \\ \operatorname{Coker}(\bar{\partial}_C) \end{bmatrix} \xrightarrow{\gamma_\rho} \operatorname{Coker}(\bar{\partial}_E) \longrightarrow 0,$$

α_ρ, β_ρ, and γ_ρ are defined similarly to the maps in Remark 3.3.

Proof. To see this consider the diagram

$$\begin{array}{ccccccc} \begin{bmatrix} \operatorname{Ker}(\bar{\partial}_{A\sharp B}) \\ \operatorname{Ker}(\bar{\partial}_C) \end{bmatrix} & \xrightarrow{\operatorname{id}} & \begin{bmatrix} \operatorname{Ker}(\bar{\partial}_{A\sharp B}) \\ \operatorname{Ker}(\bar{\partial}_C) \end{bmatrix} & \longrightarrow & \begin{bmatrix} \operatorname{Coker}(\bar{\partial}_{A\sharp B}) \\ \operatorname{Coker}(\bar{\partial}_C) \end{bmatrix} & \xrightarrow{\operatorname{id}} & \begin{bmatrix} \operatorname{Ker}(\bar{\partial}_{A\sharp B}) \\ \operatorname{Ker}(\bar{\partial}_C) \end{bmatrix} \\ \uparrow & & \downarrow & & \uparrow & & \downarrow \\ \operatorname{Ker}(\bar{\partial}_E) & \xrightarrow{\alpha_\rho} & \begin{bmatrix} \operatorname{Ker}(\bar{\partial}_A) \\ \operatorname{Ker}(\bar{\partial}_B) \\ \operatorname{Ker}(\bar{\partial}_C) \end{bmatrix} & \xrightarrow{\beta_\rho} & \begin{bmatrix} \operatorname{Coker}(\bar{\partial}_A) \\ \operatorname{Coker}(\bar{\partial}_B) \\ \operatorname{Coker}(\bar{\partial}_C) \end{bmatrix} & \xrightarrow{\gamma_\rho} & \operatorname{Coker}(\bar{\partial}_E) \end{array},$$

where the 0's at the ends of the horizontal rows are dropped as are the 0's above the two middle terms in the upper horizontal row. Completing the diagram with an arrow from the left middle entry in the lower row to the right middle entry in the upper row we find the direct sum of the gluing sequence of $\bar{\partial}_{A\sharp B}$ and the trivial sequence for $\bar{\partial}_C$. We check orientations. Let

$$(\sigma, \bar{\sigma}) \in \Lambda^{\max}(\operatorname{Ker}(\bar{\partial}_E)) \times \Lambda^{\max}(\operatorname{Coker}(\bar{\partial}_E))$$

represent an orientation pair for $\det(\bar{\partial}_E)$. Pick a wedge of vectors $\omega \wedge \gamma$ on the complement of its image in $\operatorname{Ker}(\bar{\partial}_{A\sharp B}) \oplus \operatorname{Ker}(\bar{\partial}_C)$ where ω is a wedge of vectors in $\operatorname{Ker}(\bar{\partial}_{A\sharp B})$ and γ a wedge of vectors in $\operatorname{Ker}(\bar{\partial}_C)$. Then the induced orientation on the pair

$$\left(\begin{bmatrix} \operatorname{Ker}(\bar{\partial}_{A\sharp B}) \\ \operatorname{Ker}(\bar{\partial}_C) \end{bmatrix}, \begin{bmatrix} \operatorname{Coker}(\bar{\partial}_{A\sharp B}) \\ \operatorname{Coker}(\bar{\partial}_C) \end{bmatrix} \right)$$

is represented by

$$(\sigma \wedge \omega \wedge \gamma, \bar{\gamma} \wedge \bar{\omega} \wedge \bar{\sigma}),$$

where $\bar{\alpha}$ is the image of the form α under the appropriate map in Remark 3.3 with the vectors in the opposite order, see Subsection 3.2.1. This latter orientation pair induces the orientation pair

$$(\sigma \wedge \omega \wedge \gamma \wedge \eta, \bar{\eta} \wedge \bar{\gamma} \wedge \bar{\omega} \wedge \bar{\sigma}).$$

on the pair

$$\left(\begin{bmatrix} \operatorname{Ker}(\bar{\partial}_A) \\ \operatorname{Ker}(\bar{\partial}_B) \\ \operatorname{Ker}(\bar{\partial}_C) \end{bmatrix}, \begin{bmatrix} \operatorname{Coker}(\bar{\partial}_A) \\ \operatorname{Coker}(\bar{\partial}_B) \\ \operatorname{Coker}(\bar{\partial}_C) \end{bmatrix} \right),$$

where η is any wedge of vectors in $\operatorname{Ker}(\bar{\partial}_A) \oplus \operatorname{Ker}(\bar{\partial}_B)$ on the complement of the image of ω. On the other hand the orientation pair $(\sigma, \bar{\sigma})$ induces the orientation pair

$$(\sigma \wedge \phi, \bar{\phi} \wedge \bar{\sigma})$$

on the same pair of spaces. Since ϕ and $\omega \wedge \gamma \wedge \eta$ represents the same orientation on the complement of σ if and only if $\bar{\phi}$ represents the same orientation as $\bar{\eta} \wedge \bar{\gamma} \wedge \bar{\omega}$ on the complement of $\bar{\sigma}$ we see that the orientation pairs agree and the two gluing sequences give the same orientation on $\det(\bar{\partial}_E)$. An identical argument shows that also the gluing sequence for $\bar{\partial}_A$ and $\bar{\partial}_{B\sharp C}$ induces this orientation on $\det(\bar{\partial}_E)$. $\square$

Second we consider the mixed case. Let $A\colon \partial D_m \to U(n)$, $B\colon \partial D_r \to U(n)$, and $C\colon \partial D_s \to U(n)$ be trivialized transverse boundary conditions for the $\bar{\partial}$-operator on three punctured disks. Assume that A can be glued to B at a boundary point and that B can be glued to C at puncture. Performing these two gluings we obtain a boundary condition $E\colon \partial D_{m+r+s-2} \to U(n)$. Fix orientations o_A, o_B, and o_C on $\det(\bar{\partial}_A)$, $\det(\bar{\partial}_B)$, and $\det(\bar{\partial}_C)$, respectively. Via Lemmas 3.1 and 3.4 these orientations induce orientations $o_{A\sharp B}$ and $o_{B\sharp C}$ on $\det(\bar{\partial}_{A\sharp B})$ and $\det(\bar{\partial}_{B\sharp C})$, respectively.

Lemma 3.6. *Let o'_E be the orientation on $\det(\bar{\partial}_E)$ induced via gluing (as in Lemma 3.4) from the orientations of $o_{A\sharp B}$ and o_C on $\det(\bar{\partial}_{A\sharp B})$ and $\det(\bar{\partial}_C)$, respectively. Let o''_E be the orientation on $\det(\bar{\partial}_E)$ induced via gluing (as in Lemma 3.1) from the orientations of o_A and $o_{B\sharp C}$ on $\det(\bar{\partial}_A)$ and $\det(\bar{\partial}_{B\sharp C})$, respectively. Then*

$$o'_E = o''_E = o_E,$$

where o_E is the orientation induced from the exact sequence

$$0 \longrightarrow \mathrm{Ker}(\bar{\partial}_E) \xrightarrow{\ \alpha_\rho\ } \begin{bmatrix} \mathrm{Ker}(\bar{\partial}_A) \\ \mathrm{Ker}(\bar{\partial}_B) \\ \mathrm{Ker}(\bar{\partial}_C) \end{bmatrix} \xrightarrow{\ \beta_\rho\ }$$

$$\begin{bmatrix} \mathrm{Coker}(\bar{\partial}_A) \\ \mathbb{R}^n \\ \mathrm{Coker}(\bar{\partial}_B) \\ \mathrm{Coker}(\bar{\partial}_C) \end{bmatrix} \xrightarrow{\ \gamma_\rho\ } \mathrm{Coker}(\bar{\partial}_E) \longrightarrow 0,$$

where α_ρ, β_ρ, and γ_ρ are the naturally defined maps in a simultaneous gluing process (see Remark 3.3).

Proof. The proof is similar to the proof of Lemma 3.5. $\qquad\qquad\qquad\square$

Finally we consider the case of two gluings at boundary points. Let $A\colon \partial D_m \to U(n)$, $B\colon \partial D_r \to U(n)$, and $C\colon \partial D_s \to U(n)$ be trivialized transverse boundary conditions for the $\bar{\partial}$-operator on three punctured disks. Assume that A can be glued to B at a boundary point and that B can be glued to C at a boundary point. Performing these two gluings we obtain a boundary condition $E\colon \partial D_{m+r+s} \to U(n)$. Fix orientations o_A, o_B, and o_C on $\det(\bar{\partial}_A)$, $\det(\bar{\partial}_B)$, and $\det(\bar{\partial}_C)$, respectively. Via Lemma 3.1 these orientations induce orientations $o_{A\sharp B}$ and $o_{B\sharp C}$ on $\det(\bar{\partial}_{A\sharp B})$ and $\det(\bar{\partial}_{B\sharp C})$, respectively.

Lemma 3.7. *Let o'_E be the orientation on $\det(\bar{\partial}_E)$ induced via gluing (as in Lemma 3.1) from the orientations of $o_{A\sharp B}$ and o_C on $\det(\bar{\partial}_{A\sharp B})$ and $\det(\bar{\partial}_C)$, respectively. Let o''_E be the orientation on $\det(\bar{\partial}_E)$ induced via gluing (as in Lemma 3.1) from the orientations of o_A and $o_{B\sharp C}$ on $\det(\bar{\partial}_A)$ and $\det(\bar{\partial}_{B\sharp C})$, respectively. Then*

$$o'_E = o''_E = o_E,$$

where o_E is the orientation induced from the exact sequence

$$0 \longrightarrow \mathrm{Ker}(\bar{\partial}_E) \xrightarrow{\ \alpha_\rho\ } \begin{bmatrix} \mathrm{Ker}(\bar{\partial}_A) \\ \mathrm{Ker}(\bar{\partial}_B) \\ \mathrm{Ker}(\bar{\partial}_C) \end{bmatrix} \xrightarrow{\ \beta_\rho\ }$$

$$\begin{bmatrix} \mathrm{Coker}(\bar{\partial}_A) \\ \mathbb{R}^n \\ \mathrm{Coker}(\bar{\partial}_B) \\ \mathbb{R}^n \\ \mathrm{Coker}(\bar{\partial}_C) \end{bmatrix} \xrightarrow{\ \gamma_\rho\ } \mathrm{Coker}(\bar{\partial}_E) \longrightarrow 0,$$

where α_ρ, β_ρ, and γ_ρ are naturally defined maps in a simultaneous gluing process (see Remark 3.3).

Proof. The proof is similar to the proof of Lemma 3.5. □

3.3. **Canonical orientations and capping disks.** We describe, following [12], the canonical orientation of the determinant bundle over the space of trivialized Lagrangian boundary conditions on the closed (0-punctured) disk. We also relate trivialized Lagrangian boundary conditions on a punctured disk with trivialized boundary conditions on the closed disk using capping disks.

3.3.1. *The canonical orientation of the determinant bundle over trivialized boundary conditions on the disk.* The space of trivialized Lagrangian boundary conditions over the 0-punctured disk is the space $\Omega(U(n))$ of (free) loops in $U(n)$.

Lemma 3.8. *The determinant bundle over $\Omega(U(n))$ is orientable. In fact, an orientation of $\mathbb{R}^n$ induces an orientation of this bundle.*

Proof. Consider the fibration $\Omega(U(n)) \to U(n)$ which is evaluation at $1 \in \partial D$. We find that if Y is any component of $\Omega(U(n))$ then $\pi_1(Y) = \mathbb{Z}$ since $\pi_1(U(n)) = \mathbb{Z}$ and $\pi_2(U(n)) = 0$. In fact, a generator of $\pi_1(Y)$ can be described as follows. Fix any element $A \colon \partial D \to U(n)$ in Y and a loop $B(s)$ in $U(n)$ which starts and ends at id and which generates $\pi_1(U(n))$. Then the loop $s \mapsto A \cdot B(s)$ generates $\pi_1(\Omega(U(n)))$. Let $E \to Y$ be the determinant bundle. We need to check that $w_1(E) = 0$, where w_1 is the first Stiefel-Whitney class. To see this note that the monodromy of the orientation bundle of E along the loop described is the following. If $v_1, \ldots, v_r$ is a basis in the kernel and $w_1, \ldots, w_s$ one in the cokernel of $\bar{\partial}_A$ then the monodromy is given by the orientation pairs

$$\Big(B(s)v_1 \wedge \cdots \wedge B(s)v_r, B^*(s)w_1 \wedge \cdots \wedge B^*(s)w_s \Big).$$

Hence it preserves orientation and $w_1(E) = 0$. To prove the second statement, we follow the argument in [12], §21. The trivialization A on the boundary allow us to choose a trivialization of the trivial bundle $\mathbb{C}^n$ near the boundary of ∂D such that the real plane field is constantly equal to $\mathbb{R}^n \subset \mathbb{C}^n$. Extending this trivialization inwards towards the center of the disk and stretching the neck, we eventually split off a $\mathbb{C}P^1$ with a complex vector bundle at $0 \in D$. Using gluing arguments similar to those above we find that orientations of the determinant line over the disk with constant boundary condition together with the complex orientation of the determinant line of the complex bundle over $\mathbb{C}P^1$ induces an orientation of $\det(\bar{\partial}_A)$. Since the determinant line of the $\bar{\partial}$-operator is canonically (by evaluation at any point in the boundary) identified with $\Lambda^n \mathbb{R}^n$ we get an induced orientation as claimed. □

Remark 3.9. As shown in [12] the orientations on the index space induced by different trivializations (different mod 2 if $n = 2$) are different.

Fix an orientation on $\mathbb{R}^n$.

Definition 3.10. The orientation of the determinant bundle over $\Omega(U(n))$ induced from the fixed orientation on $\mathbb{R}^n$ will be called the *canonical orientation*.

Consider two trivialized boundary conditions A and B on the 0-punctured disk as in Lemma 3.1 and construct the glued boundary condition $A \sharp B$.

Lemma 3.11. *The gluing sequence in Lemma 3.1 induces from the canonical orientations on $\det(\bar{\partial}_A)$ and on $\det(\bar{\partial}_B)$, and the orientation $(-1)^{\frac{1}{2}(n-1)(n-2)}$ times the standard orientation on $\mathbb{R}^n$, the canonical orientation on $\det(\bar{\partial}_{A\sharp B})$.*

Proof. As in the proof of Lemma 3.8 we pick special trivializations of the bundles near the boundary. Note that these trivializations glue in a natural way. When pushing inwards we obtain two complex bundles over two copies of $\mathbb{C}P^1$ sitting over the origins of the two disks glued. Using a gluing argument similar to the one in Lemma 3.1, we see that the lemma will follow as soon as it is proved for the disk with constant boundary conditions $\mathbb{R}^n \subset \mathbb{C}^n$. In this case $\mathrm{Ker}(\bar{\partial}_A)$, $\mathrm{Ker}(\bar{\partial}_B)$, and $\mathrm{Ker}(\bar{\partial}_{A\sharp B})$ are all isomorphic to $\mathbb{R}^n$ and the cokernels of all three operators are trivial. Moreover, the fixed orientation on $\mathbb{R}^n$ gives the canonical orientation on the determinants. The gluing sequence is

$$0 \longrightarrow \mathbb{R}^n \xrightarrow{\alpha_\rho} \mathbb{R}^n \oplus \mathbb{R}^n \xrightarrow{\beta_\rho} \mathbb{R}^n \longrightarrow 0,$$

where, as is easily seen, $\alpha_\rho(v) = (v, v)$. To determine β_ρ, let $w \in \mathbb{R}^n$ be thought of as an element in the kernel on the first disk. We then cut this constant function off by a cut-off function ϕ which equals 1 on the first disk and equals 0 on the second. The cokernel of the $\bar{\partial}$-operator on the strip with positive weights and constant boundary conditions is a complement of the intersections of the kernel of the L^2-pairing with constant functions. We can thus represent it as the subspace spanned by constant functions in the strip which are cut-off *outside* the supports of the cut-off functions ϕ. Now, $\bar{\partial}(\phi w) = (\frac{\partial \phi}{\partial \tau} + i \frac{\partial \phi}{\partial t})w$, and taking the L^2-inner product with the cut-off constant functions we find that the sign of $\frac{\partial \phi}{\partial \tau}$ essentially determines the map. Since the orientation of the τ-axis is from the first disk to the second we find that $\beta_\rho(w_1, w_2) = -w_1 + w_2$. It follows from our conventions in Section 3.2.1 that the standard orientation on $\mathbb{R}^n \oplus \mathbb{R}^n$ and the orientation $(-1)^{\frac{1}{2}(n-1)(n-2)}$ times the standard orientation the second $\mathbb{R}^n$ induces the standard orientation on the first $\mathbb{R}^n$. The lemma follows. $\square$

Remark 3.12. Note that it follows from the above Lemma 3.11 that picking two disks with constant boundary conditions and gluing these we get a disk with constant boundary conditions and the orientation induced on the determinant of the later from orientations on the determinants of the former two is independent of the ordering of the former two. The reason for this is that the change of order of the summands is accompanied by a change of the map β_ρ to $-\beta_\rho$. If the dimension n is even then both these changes preserve orientation and, if n is odd both reverse orientation. Thus in either case the induced orientation is not affected by the change of order.

3.3.2. *Transverse Lagrangian subspaces and unitary operators.* Let V_1 and V_2 be two transverse Lagrangian subspaces of $\mathbb{C}^n$. Let $\theta = (\theta_1, \ldots, \theta_n) \in (0, \pi)^n$ be the complex angle of (V_1, V_2) and recall that there exists canonical complex coordinates $(x_1 + iy_1, \ldots, x_n + iy_n)$ in $\mathbb{C}^n$ such that $V_1 = \mathrm{Span}(\partial_{x_1}, \ldots, \partial_{x_n})$ and $V_2 = (e^{i\theta_1}\partial_{x_1}, \ldots, e^{i\theta_n}\partial_{x_n})$. Let $Q[V_1, V_2](s)$ be the 1-parameter family of unitary transformations of $\mathbb{C}^n$ given by the matrix

$$\mathrm{Diag}(e^{i\theta_1 s}, \ldots, e^{i\theta_n s})$$

in canonical coordinates. Then $(V_1, V_2) \mapsto Q[V_1, V_2](s)$ defines a map of the space of pairs of transverse Lagrangian subspaces in $\mathbb{C}^n$ into the path space of $U(n)$. (Note that $Q[V_1, V_2]$ is independent of the choice of canonical coordinates.) The next lemma shows that this map is continuous.

Lemma 3.13. *The 1-parameter family of unitary transformations $Q[V_1, V_2](s)$ depends continuously on (V_1, V_2).*

Proof. The space of Lagrangian subspaces of $\mathbb{C}^n$ transverse to $\mathbb{R}^n$ is identified with the space of symmetric linear matrices L via

$$L \mapsto V(L) = \mathrm{Span}(L\partial_1 + i\partial_1, \ldots, L\partial_n + i\partial_n).$$

If $\cot^{-1}\colon (-\infty, \infty) \to (0, \pi)$ is the inverse of $\cot = \frac{\cos}{\sin}$ then it is easily checked that $Q[\mathbb{R}^n, V(L)](s)$ is given by the matrix

$$(3.14) \qquad\qquad\qquad e^{is\cot^{-1}(L)}$$

in the standard basis of $\mathbb{C}^n$. The lemma is a straightforward consequence of (3.14). $\qquad\square$

Recall, see Subsection 3.1.2, that we subdivided the set of pairs of transverse oriented Lagrangian subspaces into two subsets: even pairs and odd pairs. We associate to such pairs, with complex angle meeting certain conditions, 1-parameter families of unitary transformations. We define these unitary transformations by writing their matrices in canonical coordinates. Note that the extra conditions on the complex angle ensures that the transformations are independent of the choice of canonical coordinates. Let $\theta(V_1, V_2) = (\theta_1, \ldots, \theta_n)$ denote the complex angle of the ordered pair (V_1, V_2).

(**ne**) If (V_1, V_2) is even and if $\theta_1 < \theta_j$ for all $j = 2, \ldots, n$ then define

$$R_{ne}[V_1, V_2](s) = \mathrm{Diag}\left(e^{i\theta_1 s}, e^{-i(\pi - \theta_2)s}, \ldots, e^{-i(\pi - \theta_n)s}\right).$$

(**no**) If (V_1, V_2) is odd and if $\theta_1 < \theta_2 < \theta_j$, $j = 3, \ldots, n$ then define

$$R_{no}[V_1, V_2](s) = \mathrm{Diag}\left(e^{i\theta_1 s}, e^{-i(2\pi - \theta_2)s}, e^{-i(\pi - \theta_3)s}, \ldots, e^{-i(\pi - \theta_n)s}\right).$$

(**pe**) If (V_2, V_1) is even and if $\theta_n > \theta_j$, $j = 1, \ldots, n - 1$ then define

$$R_{pe}[V_1, V_2](s) = \mathrm{Diag}\left(e^{-i(2\pi - \theta_1)s}, \ldots, e^{-i(2\pi - \theta_{n-1})s}, e^{-i(\pi - \theta_n)s}\right).$$

(**po**) If (V_2, V_1) is odd and if $\theta_n > \theta_{n-1} > \theta_j$, $j = 1, \ldots, n - 2$ then define

$$R_{po}[V_1, V_2](s) = \mathrm{Diag}\left(e^{-i(2\pi - \theta_1)s}, \ldots, e^{-i(2\pi - \theta_{n-2})s}, e^{-i(\pi - \theta_{n-1})s}, e^{-i(\pi - \theta_n)s}\right).$$

Let $\hat{\pi} = (\pi, \ldots, \pi)$ and note that if the complex angle of (V_1, V_2) equals θ then the complex angle of (V_2, V_1) equals $\hat{\pi} - \theta$. It is then easily seen that if (V_1, V_2) satisfies **ne** and (V_2, V_1) satisfies **pe** then $R_{pe}[V_2, V_1](1) \circ R_{ne}[V_1, V_2](1) = \mathrm{id}$. Similarly, if (V_1, V_2) satisfies **no** and (V_2, V_1) satisfies **po** then $R_{po}[V_2, V_1](1) \circ R_{no}[V_1, V_2](1) = \mathrm{id}$.

3.3.3. *Stabilization of Lagrangian subspaces.* Let V be an oriented Lagrangian subspace of $\mathbb{C}^n$ and let $0 \le \beta < \pi$. Let $L^u(\beta)$ and $L^l(\beta)$ be the oriented Lagrangian subspaces of $\mathbb{C}^2$, with standard coordinates $(x_1 + iy_1, x_2 + iy_2)$, given by the orienting basis

$$L^u(\beta) = \left(e^{i\frac{\beta}{2}}\partial_{x_1}, e^{i\beta}\partial_{x_2}\right),$$

$$L^l(\beta) = \left(e^{-i\frac{\beta}{2}}\partial_{x_1}, e^{-i\beta}\partial_{x_2}\right),$$

Define the *upper β-stabilization* of V to be the oriented Lagrangian subspace V^u of $\mathbb{C}^{n+2} = \mathbb{C}^n \times \mathbb{C}^2$ given by

$$V^u(\beta) = V \times L^u(\beta).$$

Define the *lower β-stabilization* of V to be the oriented Lagrangian subspace V^l of $\mathbb{C}^{n+2} = \mathbb{C}^n \times \mathbb{C}^2$ given by

$$V^l(\beta) = V \times L^l(\beta).$$

Let $\left(V_1(\lambda), V_2(\lambda)\right)$, $\lambda \in \Lambda$ be a continuous family of transverse Lagrangian subspaces parameterized by a compact space Λ. Let $\theta(\lambda) = \left(\theta_1(\lambda), \ldots, \theta_n(\lambda)\right)$ be the complex angle of $\left(V_1(\lambda), V_2(\lambda)\right)$. By compactness of Λ there exists $\beta > 0$ such that $2\beta < \theta_j(\lambda)$ and $\pi - 2\beta > \theta_j(\lambda)$ for all $\lambda \in \Lambda$ and $j = 1, \ldots, n$. Fix such a $\beta > 0$ and let $\tilde{V}_1(\lambda) = [V_1(\lambda)]^u(\beta)$ and $\tilde{V}_2(\lambda) =$

$[V_2(\lambda)]^l(\beta)$. Note that $\big(\tilde{V}_1(\lambda), \tilde{V}_2(\lambda)\big)$ is even (odd) if and only if $\big(V_1(\lambda), V_2(\lambda)\big)$ is even (odd). Moreover, by the choice of β, $\big(\tilde{V}_1(\lambda), \tilde{V}_2(\lambda)\big)$ satisfies the condition **ne** (**no**) if $\big(V_2(\lambda), V_1(\lambda)\big)$ is even (odd) and $\big(\tilde{V}_2(\lambda), \tilde{V}_1(\lambda)\big)$ satisfies the condition **pe** (**po**) if $\big(V_1(\lambda), V_2(\lambda)\big)$ is even (odd), for all $\lambda \in \Lambda$. Thus we can construct the corresponding Λ-families of unitary operators.

Lemma 3.14. *Let $\big(V_1(\lambda), V_2(\lambda)\big)$ and $\beta > 0$ be as above. If $\big(V_1(\lambda), V_2(\lambda)\big)$ is even then the families of unitary operators $R_{ne}[\tilde{V}_1(\lambda), \tilde{V}_2(\lambda)](s)$ and $R_{pe}[\tilde{V}_2(\lambda), \tilde{V}_1(\lambda)](s)$ depend continuously on $\lambda \in \Lambda$. If $\big(V_1(\lambda), V_2(\lambda)\big)$ is odd then the families of unitary operators $R_{no}[\tilde{V}_1(\lambda), \tilde{V}_2(\lambda)](s)$ and $R_{po}[\tilde{V}_2(\lambda), \tilde{V}_1(\lambda)](s)$ depend continuously on $\lambda \in \Lambda$.*

Proof. This is a straightforward consequence of (3.14). □

3.3.4. *Determinant bundles over stabilized Lagrangian subspaces.* Let $\big(V_1(\lambda), V_2(\lambda)\big)$, $\lambda \in \Lambda$ be a continuous family of transverse Lagrangian subspaces of $\mathbb{C}^n$ parameterized by a compact simply connected space Λ. Fix $\beta > 0$ small enough and consider the stabilized family $\big(\tilde{V}_1(\lambda), \tilde{V}_2(\lambda)\big)$ of transverse Lagrangian subspaces in $\mathbb{C}^{n+2}$. Assume that $\tilde{V}_1(\lambda)$ and $\tilde{V}_2(\lambda)$ are equipped with positively oriented frames $X_1(\lambda)$ and $X_2(\lambda)$ which vary continuously with $\lambda \in \Lambda$. We associate to this family two families of trivialized Lagrangian boundary conditions on the 1-punctured disk. To simplify notation, if $\big(V_1(\lambda), V_2(\lambda)\big)$ is even then let $* = e$ and if $\big(V_1(\lambda), V_2(\lambda)\big)$ is odd then let $* = o$. Note that

$$R_{n*}[\tilde{V}_1(\lambda), \tilde{V}_2(\lambda)](1)X_1(\lambda)$$

is a framing of $\tilde{V}_2(\lambda)$. Hence there exists $\alpha(\lambda) \in SO(n+2)$ such that

$$R_{n*}[\tilde{V}_1(\lambda), \tilde{V}_2(\lambda)](1)X_1(\lambda) = X_2(\lambda) \cdot \alpha(\lambda).$$

Since Λ is simply connected the map $\alpha \colon \Lambda \to SO(n+2)$ lifts to a map $\tilde{\alpha} \colon \Lambda \to PSO(n+2)$, where $PSO(n+2)$ is the space of paths in $SO(n+2)$ with initial endpoint at the identity matrix and which projects to $SO(n+2)$ by evaluation at the final endpoint. Pick such a lift. Identify ∂D_1 with $[0,1]$. Define two families of trivialized boundary conditions for the $\bar{\partial}$-operator on D_1 as follows

$$(3.15) \qquad A_{n*}[\lambda](s) = R_{n*}[\tilde{V}_1(\lambda), \tilde{V}_2(\lambda)]X_1(\lambda) \cdot \tilde{\alpha}[\lambda](s),$$

$$(3.16) \qquad A_{p*}[\lambda](s) = R_{p*}[\tilde{V}_2(\lambda), \tilde{V}_1(\lambda)]X_2(\lambda) \cdot \tilde{\alpha}^{-1}[\lambda](s),$$

where $\tilde{\alpha}^{-1}[\lambda](s)$ is the inverse of the matrix $\tilde{\alpha}[\lambda](s)$. Let p be the puncture on D_1 and note that $A_{n*}[\lambda](p^{\pm}) = A_{p*}[\lambda](p^{\mp})$ so that these boundary conditions can be glued. Let $\bar{\partial}_{n*,\lambda}$ and $\bar{\partial}_{p*,\lambda}$ denote the $\bar{\partial}$-operators with boundary conditions $A_{n*}[\lambda]$ and $A_{p*}[\lambda]$, respectively. Then the bundles $E_{n*} = \det(\bar{\partial}_{n*,\lambda}) \to \Lambda$ and $E_{p*} = \det(\bar{\partial}_{p*,\lambda}) \to \Lambda$ are orientable since Λ is simply connected. We orient them as follows. Let $0 \in \Lambda$. Pick an orientation of $\det(\bar{\partial}_{n*,0})$. Together with the canonical orientation of the $\bar{\partial}$-operator on the 0-punctured disk and Lemma 3.4 this orientation determines an orientation on $\det(\bar{\partial}_{p*,0})$. Since the bundles E_{n*} and E_{p*} are orientable the orientations of $\det(\bar{\partial}_{n*,0})$ and $\det(\bar{\partial}_{p*,0})$ induce orientations on $\det(\bar{\partial}_{n*,\lambda})$ and $\det(\bar{\partial}_{p*,\lambda})$, respectively, for any $\lambda \in \Lambda$.

Lemma 3.15. *The orientations on $\det(\bar{\partial}_{n*,\lambda})$ and $\det(\bar{\partial}_{p*,\lambda})$ as defined above glue to the canonical orientation on the 0-punctured disk.*

Proof. After adding a finite dimensional vector space we may assume that all operators are surjective. The lemma then follows from properties of the direct sum operation on vector bundles. □

We call the operators $\bar{\partial}_{ne,\lambda}$, $\bar{\partial}_{pe,\lambda}$, $\bar{\partial}_{no,\lambda}$, and $\bar{\partial}_{po,\lambda}$ which arises as above *capping operators* and we call an orientation pair on E_{n*}, E_{p*} with the properties above a pair of *capping orientations*.

3.3.5. *Capping orientations of trivialized boundary conditions.* Let Λ be any compact simply connected space and let $\left(V_1^j(\lambda), V_2^j(\lambda)\right)$, $j = 0, \ldots, m$ be Λ-families of transverse Lagrangian subspaces. Construct as in the previous section the stabilizations $\left(\tilde{V}_1^j(\lambda), \tilde{V}_2^j(\lambda)\right)$ and assume that these families are equipped with positively oriented frames $X_1(\lambda)$ and $X_2(\lambda)$, respectively. Construct the families $\bar{\partial}_{ne,\lambda}^j$, $\bar{\partial}_{pe,\lambda}^j$, $\bar{\partial}_{no,\lambda}^j$, and $\bar{\partial}_{po,\lambda}^j$ of operators on the 1-punctured disks with oriented determinant bundles as there. Let X be any topological space and consider an $(X \times \Lambda)$-family of oriented Lagrangian boundary conditions $a[x,\lambda] \colon \partial D_{m+1} \to \mathrm{Lag}(n)$, $(x,\lambda) \in X \times \Lambda$ on the $(m+1)$-punctured disk with the following properties. For all (x,λ), if $j \geq 1$ then $a[x,\lambda](p_j^+) = V_1^1(\lambda)$, $a[x,\lambda](p_j^-) = V_2^j(\lambda)$, $a[x,\lambda](p_0^+) = V_2^0(\lambda)$, and $a[x,\lambda](p_0^-) = V_1^0(\lambda)$. Let $E \to X \times \Lambda$ denote the determinant bundle of $\bar{\partial}_{a[x,\lambda]}$. We construct, under some additional trivialization conditions, the *capping orientation* of E in the following way. First stabilize the family $a[x,\lambda] \colon \partial D_m \to \mathrm{Lag}(n)$ to $\tilde{a}[x,\lambda] \colon \partial D_m \to \mathrm{Lag}(n+2)$ by letting

$$\tilde{a}[x,\lambda](\zeta) = a[x,\lambda](\zeta) \times \mathrm{Span}\left(e^{i\frac{\gamma}{2}(\zeta)}\partial_{x_1}, e^{i\gamma(\zeta)}\partial_{x_2}\right), \quad \zeta \in \partial D_m,$$

where $\mathbb{C}^{n+2} = \mathbb{C}^n \times \mathbb{C}^2$, $\mathbb{C}^2 = \{(x_1 + iy_1, x_2 + iy_2)\}$, and where $\gamma \colon \partial D_m \to [-\beta, \beta]$ satisfies $\gamma(\zeta) = \beta$ in neighborhoods of p_j^+, $j \geq 1$ and p_0^-, and $\gamma(\zeta) = -\beta$ in neighborhoods of p_j^- and p_0^+. Noting that the $\bar{\partial}$-operator on D_m with boundary conditions

$$\zeta \mapsto \mathrm{Diag}\left(e^{i\frac{\gamma}{2}(\zeta)}\partial_{x_1}, e^{i\gamma(\zeta)}\partial_{x_2}\right)$$

has both trivial kernel and trivial cokernel we find that the kernels and cokernels of $\bar{\partial}_{a[x,\lambda]}$ and $\bar{\partial}_{\tilde{a}[x,\lambda]}$ are canonically isomorphic. Thus to orient the determinant bundle of $\bar{\partial}_{a[x,\lambda]}$ it suffices to orient the determinant bundle of $\bar{\partial}_{\tilde{a}[x,\lambda]}$. We find such an orientation in the case when the boundary conditions $\tilde{a}[x,\lambda]$ are equipped with a certain type of trivialization. Thus, assume that the boundary conditions $\tilde{a}[x,\lambda] \colon \partial D_m \to \mathrm{Lag}(n+2)$ are trivialized, i.e. they are represented by $\tilde{A}[x,\lambda] \colon \partial D_m \to U(n+2)$. Assume moreover that this trivialization satisfies the following conditions for all (x,λ), if $j \geq 1$ then $\tilde{A}[x,\lambda](p_j^+) = X_1^j(\lambda)$, $\tilde{A}[x,\lambda](p_j^-) = X_2^j(\lambda)$, $\tilde{A}[x,\lambda](p_0^+) = X_2^0(\lambda)$, and $\tilde{A}[x,\lambda](p_0^-) = X_1^0(\lambda)$.

Glue to $\bar{\partial}_{\tilde{A}[x,\lambda]}$ first the capping operator $\bar{\partial}_{pe,\lambda}^0$ ($\bar{\partial}_{po,\lambda}^0$) at p_0 if the puncture p_0 is even (odd). Then glue to $\bar{\partial}_{\tilde{A}[x,\lambda]}$ the operators $\bar{\partial}_{ne,\lambda}^j$ ($\bar{\partial}_{ne,\lambda}^j$) to $\tilde{A}[x,\lambda]$ at its even (odd) negative puncture p_j, $j \geq 1$, *in the order opposite* to that induced by the boundary orientation of ∂D_{m+1}. (We glue in the opposite order so that the Leibniz rule works out appropriately, see Section 4.) We obtain in this way a trivialized boundary condition $\hat{A}[x,\lambda] \colon \partial D \to U(n+2)$ on the closed disk. Lemma 3.4 and repeated application of Lemma 3.5 gives the gluing sequence, where we write $\bar{\partial}_{q^\pm,\lambda}$ for the capping operator at the puncture q at $\lambda \in \Lambda$ and where the sign indicates

the sign of the puncture,

(3.17)

$$0 \longrightarrow \mathrm{Ker}(\bar{\partial}_{\hat{A}[x,\lambda]}) \longrightarrow \begin{bmatrix} \mathrm{Ker}(\bar{\partial}_{p_1^-,\lambda}) \\ \vdots \\ \mathrm{Ker}(\bar{\partial}_{p_m^-,\lambda}) \\ \mathrm{Ker}(\bar{\partial}_{p_0^+,\lambda}) \\ \mathrm{Ker}(\bar{\partial}_{\tilde{A}[x,\lambda]}) \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} \mathrm{Coker}(\bar{\partial}_{p_1^-,\lambda}) \\ \vdots \\ \mathrm{Coker}(\bar{\partial}_{p_m^-,\lambda}) \\ \mathrm{Coker}(\bar{\partial}_{p_0^+,\lambda}) \\ \mathrm{Coker}(\bar{\partial}_{\tilde{A}[x,\lambda]}) \end{bmatrix} \longrightarrow \mathrm{Coker}(\bar{\partial}_{\hat{A}[x,\lambda]}) \longrightarrow 0.$$

We give the determinant of $\bar{\partial}_{\tilde{A}[x,\lambda]}$ the unique orientation o which together with the chosen orientations for the capping disks, via the the gluing sequence (3.17), give the canonical orientation on $\det(\bar{\partial}_{\hat{A}[x,\lambda]})$. It is clear that the orientation so defined gives an orientation of the determinant bundle over $\det(\bar{\partial}_{\tilde{A}[x,\lambda]}) \to X \times \Lambda$ and thus by the above mentioned isomorphism also the bundle $E = \det(\bar{\partial}_{a[x,\lambda]})$ gets oriented.

Definition 3.16. We call the orientation of the determinant bundle $E \to X \times \Lambda$ the *capping orientation*.

Note that the same construction can be applied when $X \times \Lambda$ is replaced by a locally trivial fibration $Y \to \Lambda$.

3.3.6. *Kernels and cokernels of capping operators.* Let (V_1, V_2) be a pair of transverse Lagrangian subspaces in $\mathbb{C}^{n-2}$ and let $(\tilde{V}_1, \tilde{V}_2)$ in $\mathbb{C}^n$ be its stabilization. Note that in canonical coordinates of $(\tilde{V}_1, \tilde{V}_2)$ the boundary conditions of the capping operators constructed from $(\tilde{V}_1, \tilde{V}_2)$ are split and we may determine the dimensions of the kernel and cokernel from properties of the classical Riemann-Hilbert problem. To this end let $\theta = (\theta_1, \ldots, \theta_n)$, $0 < \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n < \pi$, be the complex angle of $(\tilde{V}_1, \tilde{V}_2)$. We think of the 1-punctured disk D_1 as of the unit disk in $\mathbb{C}$ punctured at 1 boundary parameterized by e^{is}, $0 \leq s \leq 2\pi$.

- The *negative even* boundary condition is, in canonical coordinates, given by

$$s \mapsto \mathrm{Diag}\left(e^{i\theta_1 \frac{s}{2\pi}}, e^{-i(\pi-\theta_2)\frac{s}{2\pi}}, \ldots, e^{-i(\pi-\theta_n)\frac{s}{2\pi}}\right).$$

Thus,

$$\mathrm{Index}(\bar{\partial}_{ne}) = n - (n-1) = 1,$$
$$\dim(\mathrm{Ker}(\bar{\partial}_{ne})) = 1,$$
$$\dim(\mathrm{Coker}(\bar{\partial}_{ne})) = 0.$$

The kernel is spanned by the function $w(z) = (w_1(z), \ldots, w_n(z))$, where

$$w_1(z) = (i(z-1))^{\frac{\theta_1}{\pi}},$$
$$w_j(z) = 0, \quad j > 1.$$

- The *negative odd* boundary condition is, in canonical coordinates, given by

$$s \mapsto \mathrm{Diag}\left(e^{i\theta_1 \frac{s}{2\pi}}, e^{-i(2\pi-\theta_2)\frac{s}{2\pi}}, e^{-i(\pi-\theta_3)\frac{s}{2\pi}}, \ldots, e^{-i(\pi-\theta_n)\frac{s}{2\pi}}\right).$$

Thus,

$$\text{Index}(\bar{\partial}_{no}) = n - n = 0,$$
$$\dim(\text{Ker}(\bar{\partial}_{no})) = 1,$$
$$\dim(\text{Coker}(\bar{\partial}_{no})) = 1.$$

The kernel spanned by the function $w(z) = (w_1(z), \ldots, w_n(z))$, where

$$w_1(z) = (i(z-1))^{\frac{\theta_1}{\pi}},$$
$$w_j(z) = 0, \quad j > 1.$$

The cokernel is spanned by the function $u(z) = (u_1(z), \ldots, u_n(z))$, where

$$u_2(z) = (i(\bar{z}-1))^{\frac{\theta_2}{\pi}},$$
$$u_j(z) = 0, \quad j \neq 2,$$

where we view this function as a linear functional on the target space of $\bar{\partial}$ via the L^2-pairing and where the cokernel is a one dimensional subspace complementary to the kernel of this functional.

- The *positive even* boundary condition is, in canonical coordinates, given by

$$s \mapsto \text{Diag}\left(e^{-i(2\pi - \theta_1)\frac{s}{2\pi}}, \ldots, e^{-i(2\pi - \theta_{n-1})\frac{s}{2\pi}}, e^{-i(\pi - \theta_n)\frac{s}{2\pi}}\right).$$

Thus,

$$\text{Index}(\bar{\partial}_{ne}) = n - 2(n-1) - 1 = 1 - n,$$
$$\dim(\text{Ker}(\bar{\partial}_{ne})) = 0,$$
$$\dim(\text{Coker}(\bar{\partial}_{ne})) = n - 1.$$

The cokernel is spanned by the functions $u(z) = (u_1(z), \ldots, u_n(z))$, where for $k \neq n$

$$u_k(z) = (i(\bar{z}-1))^{\frac{\theta_k}{\pi}},$$
$$u_j(z) = 0, \quad j \neq k.$$

- The *positive odd* boundary condition is, in canonical coordinates, given by

$$s \mapsto \text{Diag}\left(e^{-i(2\pi - \theta_1)\frac{s}{2\pi}}, \ldots, e^{-i(2\pi - \theta_{n-2})\frac{s}{2\pi}}, e^{-i(\pi - \theta_{n-1})\frac{s}{2\pi}}, e^{-i(\pi - \theta_n)\frac{s}{2\pi}}\right).$$

Thus,

$$\text{Index}(\bar{\partial}_{ne}) = n - 2(n-2) - 2 = 2 - n,$$
$$\dim(\text{Ker}(\bar{\partial}_{ne})) = 0,$$
$$\dim(\text{Coker}(\bar{\partial}_{ne})) = n - 2.$$

The cokernel is spanned by the functions $u(z) = (u_1(z), \ldots, u_n(z))$, where for $k < n-1$

$$u_k(z) = (i(\bar{z}-1))^{\frac{\theta_k}{\pi}},$$
$$u_j(z) = 0, \quad j \neq k.$$

3.4. **Orientations of moduli spaces.** We orient moduli spaces of holomorphic disks with boundary on a generic Legendrian submanifold equipped with a spin structure. The orientation is obtained by comparing capping orientations with fixed orientations of spaces of automorphisms and of conformal structures.

3.4.1. *Orientations of spaces of automorphisms and conformal structures.* Let D_{m+1} denote the unit disk $D \subset \mathbb{C}$ with 1 positive, and m negative punctures on the boundary, $m \geq 1$. Let p_0 be the positive puncture and let $\{p_1, \ldots, p_m\}$ be the negative punctures. As mentioned above the positive puncture p_0 and the orientation of ∂D induces an ordering of the negative punctures $(p_1, p_2, \ldots, p_m)$. Let $\mathcal{C}_m$ denote the space of conformal structures on D_m. If $m \leq 3$ then $\mathcal{C}_m$ is a one-point space. For $m < 3$ let $\mathcal{A}_m$ denote the group of conformal automorphisms. We orient $\mathcal{C}_{m+1}$, $m > 2$ in the following way. Let D_m with conformal structure κ be represented by a disk with positive puncture p_0 and ordered negative punctures $(p_1, \ldots, p_m)$. Then any conformal structure κ' in a neighborhood of κ can be represented uniquely by a disk with positive puncture at p_0 its first two negative punctures at p_1 and p_2 and with the rest of its negative punctures at $(p'_3, \ldots, p'_m)$. Thus, the tangent space of $\mathcal{C}_{m+1}$ at κ is identified with the direct sum of the tangent spaces of ∂D at $p_3, \ldots, p_m$. We orient $\mathcal{C}_m$ by declaring the oriented basis

$$\left\{ \partial_{p_3} \oplus 0 \cdots \oplus 0, \ 0 \oplus \partial_{p_2} \oplus 0 \cdots \oplus 0, \ \ldots, 0 \oplus \cdots \oplus 0 \oplus \partial_{p_m} \right\},$$

where ∂_{p_j} is the positive unit tangent to ∂D at p_j, to be positive basis in $T_\kappa \mathcal{C}_m$. We orient the one-point space $\mathcal{C}_3$ by declaring it positively oriented. Next consider $\mathcal{A}_1$ and $\mathcal{A}_2$. To orient $\mathcal{A}_1$, consider D_1 punctured at p. Pick two points q_1 and q_2 in ∂D_1 a small distance from p. The tangent space of $\mathcal{A}_1$ at the identity is the 2-dimensional space of holomorphic vector fields on D tangent to ∂D along the boundary and vanishing at p. Evaluation of such a vector fields at q_1 and q_2 gives a map

$$T_{\mathrm{id}} \mathcal{A}_1 \to T_{q_1} D \oplus T_{q_2} D.$$

We use this map to orient $T_{\mathrm{id}} \mathcal{A}_1$ and the group structure of $\mathcal{A}_1$ to orient $\mathcal{A}_1$. To orient $\mathcal{A}_2$, consider D_2 as D with positive puncture at 1 and negative at -1. Pick a point q in the *lower* hemisphere of the two into which ∂D is subdivided by -1 and 1 and orient $T_{\mathrm{id}} \mathcal{A}_1$ by evaluation at q as above.

3.4.2. *Stably trivialized boundary conditions and Legendrian submanifolds with spin structures.* Let M be an orientable manifold of dimension n. Let $\tilde{T}M = TM \oplus \mathbb{R}^2$ denote the stabilized tangent bundle of M, where $\mathbb{R}^2$ denotes the trivial bundle over M. Fix some triangulation of M. Then a spin-structure on M can be viewed as a trivialization of $\tilde{T}M$ restricted to the 1-skeleton that extends to the 2-skeleton. Note that the extension to the 2-skeleton is homotopically unique if it exists since $\pi_2(SO(n+2)) = 0$. For the same reason, any trivialization over the 2-skeleton automatically extends over the 3-skeleton but this extension is in general not homotopically unique. Let L be an oriented manifold equipped with a spin structure and let Λ be a compact simply connected space. (In fact, for our applications $\Lambda = \{\text{point}\}$ and $\Lambda = [0, 1]$ are sufficient.) Let $\Phi_\lambda \colon L \to \mathbb{C}^n \times \mathbb{R}$, $\lambda \in \Lambda$ be a family of *chord generic* Legendrian embeddings. The chord genericity implies that we get a continuous family of Reeb chords $c_1(\lambda), \ldots, c_k(\lambda)$ of Φ_λ and that their endpoints vary continuously with λ in L. Fix $0 \in \Lambda$ and choose a family of diffeomorphisms $\phi_\lambda \colon L \to L$ such that $\phi_\lambda(c_j(0)^\pm) = c_j(\lambda)^\pm$ for all $j = 1, \ldots, k$. For each Reeb chord $c(0)$ of Φ_0, fix a capping path $\gamma_{c(0)} \subset L$ connecting its upper end point to the lower one, see [5], Section 2.3. Fix a triangulation $\Delta = \Delta^{(0)} \cup \cdots \cup \Delta^{(n)}$ of L, where $\Delta^{(j)}$ denotes the j-skeleton of Δ, such that each Reeb chord endpoint $c_j^\pm(0)$ lies in $\Delta^{(0)}$ and each capping path lies in $\Delta^{(1)}$. As mentioned above the spin structure gives a trivialization of the restriction of $\tilde{T}L$ to $\Delta^{(3)}$. Fix such a trivialization. As $\lambda \in \Lambda$ varies we move the triangulation and capping paths by ϕ_λ and the trivialization by the bundle isomorphism $\tilde{T}M \to \tilde{T}M$ covering ϕ_λ given by

$$\begin{pmatrix} d\phi_\lambda & 0 \\ 0 & \mathrm{id} \end{pmatrix}.$$

Using the notation from [6], Section 4, let $\mathcal{W}_{A,\Lambda}(a; b_1, \ldots, b_m)$ denote the space of candidate maps. Such a trivialization enables us to construct the capping orientation of the determinant bundle over the space of linearized boundary conditions over $\mathcal{W}_{A;\Lambda}(a; b_1, \ldots, b_m)$ in the following way. Consider first the trivial $\mathbb{C}^n$-bundle over L which is the the pull-back $(\Pi_{\mathbb{C}} \circ \Phi_\lambda)^* T\mathbb{C}^n$ and its stabilization $\mathbb{C}^n \times \mathbb{C}^2$. Associate to each point $q \in L$ the Lagrangian subspace

$$\Pi_{\mathbb{C}} \circ d\Phi_\lambda(T_p L) \oplus \mathrm{Span}\left(e^{i\frac{\alpha_\lambda(p)}{2}} \partial_{x_1}, e^{i\alpha_\lambda(p)} \partial_{x_2} \right),$$

where $\alpha_\lambda \colon L \to [-\beta, \beta]$ is a family of smooth functions such that $\alpha_\lambda(c_j^{\pm}(\lambda)) = \pm\beta$ for all Reeb chords c_j and where β is such that 2β is smaller than any component of any complex angle at any Reeb chord and $\pi - 2\beta$ is larger than such components. Let $(w, h, \kappa) \in \mathcal{W}_{A,\Lambda}(a; b_1, \ldots, b_m)$. Then the restriction of (w, h) to the part of the boundary of D_{m+1} lying between p_j and p_{j+1} is a path in L connecting two Reeb chord endpoints and we obtain from the field of Lagrangian subspaces just defined a family of Lagrangian subspaces over $\mathcal{W}_{A;\Lambda}(a; b_1, \ldots, b_m)$ which is the stabilization of the tangent plane family and which satisfies the conditions in Section 3.3.5. Pick a homotopy, fixing endpoints, of this path to a path which lies in $\Delta^{(1)}$. Then the trivialization η of $\tilde{T}L$ along $\Delta^{(1)}$ induces a trivialization of $(w, h)^* \tilde{T}L$ on the corresponding parts. This in turn induces a trivialization of the stabilized Lagrangian boundary condition and we define the capping orientation of the determinant bundle of the linearized $\bar{\partial}$-operator over $\mathcal{W}_{A,\Lambda}(a; b_1, \ldots, b_m)$ as in Definition 3.16. We must check that this orientation is well-defined. Choosing a different homotopy to some path in $\Delta^{(1)}$, the two end paths in $\Delta^{(1)}$ can be connected with a homotopy in $\Delta^{(2)}$. Using the capping orientation over this homotopy proves that the orientation is well-defined. We call also the orientation of $\det(\bar{\partial}_\kappa)$ over $\mathcal{W}_A(a; b_1, \ldots, b_m)$ the *capping orientation*.

3.4.3. *Moduli space orientations.* Using the following lemma we orient all moduli spaces of holomorphic disks. As in [6] we let Γ denote the full $\bar{\partial}$-operator.

Lemma 3.17. *Let $d\Gamma$ denote the full linearization of the $\bar{\partial}$-operator at some holomorphic $(w, h, \kappa) \in \mathcal{W}_A(a; b_1, \ldots, b_m)$. The determinant bundle over $\mathcal{W}_A(a; b_1, \ldots, b_m)$ with fiber $\det(d\Gamma)$ is canonically isomorphic to the tensor product of the determinant bundle of $\bar{\partial}_\kappa$ with the highest exterior power of the tangent bundle to $\mathcal{C}_{m+1}$.*

Proof. Let $(w, h, \kappa) \in \mathcal{W}_A$. We must show that (at (w, h, κ))

$$\Lambda^{\max} \mathrm{Ker}(d\Gamma) \otimes (\Lambda^{\max} \mathrm{Coker}(d\Gamma))^* =$$
$$\Lambda^{\max} \mathrm{Ker}(\bar{\partial}_\kappa) \otimes \left(\Lambda^{\max} \mathrm{Coker}(\bar{\partial}_\kappa) \right)^* \otimes \Lambda^{\max} T\mathcal{C}_{m+1}.$$

Note that

$$T\mathcal{W}_A = T\mathcal{W}_A(\kappa) \oplus T\mathcal{C}_{m+1} = V \oplus \mathbb{R}^N,$$

and that

$$d\Gamma = \bar{\partial}_\kappa \oplus \psi,$$

for some map

$$\psi \colon T_\kappa \mathcal{C}_{m+1} = \mathbb{R}^N \to \mathcal{H}_1(T^{*0,1} D_{m+1} \otimes \mathbb{C}^n) = W.$$

Let $\bar{\psi} \colon \mathbb{R}^N \to \mathrm{Coker}(\bar{\partial}_\kappa)$ be the map induced by projection. Then the following sequence is exact

$$0 \to \mathrm{Ker}(\bar{\psi}) \to \mathbb{R}^N \to \mathrm{Coker}(\bar{\partial}_\kappa) \to \mathrm{Coker}(\bar{\psi}) \to 0,$$

and induces a canonical isomorphism

$$\Lambda^{\max} \mathrm{Ker}(\bar{\psi}) \otimes \Lambda^{\max} \mathrm{Coker}(\bar{\partial}_\kappa) = \Lambda^{\max} \mathbb{R}^N \otimes \Lambda^{\max} \mathrm{Coker}(\bar{\psi}).$$

On the other hand $\mathrm{Coker}(d\Gamma) = \mathrm{Coker}(\bar{\psi})$ and $\mathrm{Ker}(d\Gamma) = \mathrm{Ker}(\bar{\partial}_\kappa) \oplus \mathrm{Ker}(\bar{\psi})$. Hence

$$\Lambda^{\max} \mathrm{Ker}(d\Gamma) \otimes (\Lambda^{\max} \mathrm{Coker}(d\Gamma))^* =$$
$$\Lambda^{\max} \mathrm{Ker}(\bar{\partial}_\kappa) \otimes \Lambda^{\max} \mathrm{Ker}(\bar{\psi}) \otimes (\Lambda^{\max} \mathrm{Coker}(\bar{\psi}))^* =$$
$$\Lambda^{\max} \mathrm{Ker}(\bar{\partial}_\kappa) \otimes (\Lambda^{\max} \mathrm{Coker}(\bar{\partial}_\kappa))^* \otimes \Lambda^{\max} \mathbb{R}^N.$$

$\square$

Let $(w, h, \kappa) \in \mathcal{W}_A(a; b_1, \ldots, b_m)$ be a transversely cut out holomorphic disk. Using Lemma 3.17 we orient the moduli space to which (w, h, κ) belongs as follows.

(a) If $m \leq 1$ then $d\Gamma$ simply agrees with the ordinary $\bar{\partial}$-operator and the transversality condition implies that this operator is surjective. Moreover, $v \mapsto dw \cdot v$ for $v \in T\mathcal{A}_{m+1}$ gives an injection $T\mathcal{A}_{m+1} \subset \mathrm{Ker}(\bar{\partial})$. The quotient $\mathrm{Ker}(\bar{\partial})/T\mathcal{A}_{m+1}$ can be identified with the tangent space to the moduli space $T\mathcal{M}$. The capping orientation on $\det(\bar{\partial})$ together with the orientation on $\mathcal{A}_{m+1}$ thus gives an orientation of the moduli space.

(b) If $m \geq 2$ then an orientation of $\det(\bar{\partial}_\kappa)$ and $T\mathcal{C}_{m+1}$ together give an orientation of $\det(d\Gamma)$. The surjectivity assumption implies that $\mathrm{Coker}(d\Gamma)$ is trivial hence we get an orientation on $\mathrm{Ker}(d\Gamma)$ which is the tangent space of the moduli space.

Remark 3.18. An oriented connected 0-dimensional manifold is a point with a sign. The above definition says that, to get the sign of a rigid disk we compare the capping orientation of the kernel/cokernel of the $\bar{\partial}_\kappa$-operator with the orientation on $\mathcal{A}_{m+1}$ or $\mathcal{C}_{m+1}$ depending on the number of punctures m.

4. Legendrian contact homology over $\mathbb{Z}$

In this section we associate to any Legendrian submanifold $L \subset \mathbb{C}^n \times \mathbb{R}$ which is equipped with a spin structure a graded algebra $\mathcal{A}(L)$ over the group ring $\mathbb{Z}[H_1(L)]$ if L is connected and over $\mathbb{Z}$ otherwise. We define a map $\partial \colon \mathcal{A}(L) \to \mathcal{A}(L)$ and prove that it is a differential. With this established we prove that the stable tame isomorphism class of the differential graded algebra $(\mathcal{A}(L), \partial)$ remains invariant under Legendrian isotopies. This implies in particular that the contact homology $\mathrm{Ker}(\partial)/\mathrm{Im}(\partial)$ is a Legendrian isotopy invariant. We then show how the differential ∂ depends on the particular spin structure on L and in the final subsection discuss the relation of our approach to contact homology over $\mathbb{Z}[H_1(S^1)] = \mathbb{Z}[t, t^{-1}]$ for Legendrian 1-knots with the completely combinatorial approach taken in [9].

4.1. **The algebra and its differential.** Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be (an admissible chord generic) oriented connected Legendrian submanifold. Define $\mathcal{A}(L)$ as the free associative algebra over $\mathbb{Z}[H_1(L)]$ generated by the Reeb chords of L. That is

$$\mathcal{A}(L) = \mathbb{Z}[H_1(L)]\langle c_1, \ldots, c_m \rangle.$$

Recall from [5] (also see Section 2 above) that each generator c comes equipped with a grading $|c|$ and a capping path γ_i. Elements $A \in H_1(L)$ have gradings $|A|$. So, $\mathcal{A}(L)$ is a $\mathbb{Z}$-graded algebra. We note that if L is an orientable manifold then $|A|$ is even for any $A \in H_1(L)$.

If the Legendrian submanifold L is not connected (see also [13, 14]) then we will use a simpler version of the theory: we let

$$\mathcal{A}(L) = \mathbb{Z}\langle c_1, \ldots, c_m \rangle.$$

In this case the algebra $\mathcal{A}(L)$ has a natural $\mathbb{Z}_2$-grading. There is also a relative $\mathbb{Z}/m(L)\mathbb{Z}$-grading which we will not discuss here. (Note that the orientability implies that the Maslov number is even, see also Remark 4.4). In the connected case this simpler version corresponds

to setting all $A \in H_1(L)$ to 1 and reducing the grading modulo 2. Assume that L is equipped with a spin structure. Define the *differential*

$$\partial \colon \mathcal{A}(L) \to \mathcal{A}(L)$$

by requiring that it is linear over $\mathbb{Z}[H_1(L)]$ (over $\mathbb{Z}$ in the disconnected case), that it satisfies the graded Leibniz rule on products of monomials

(4.1) $$\partial(\alpha\beta) = (\partial\alpha)\beta + (-1)^{|\alpha|}\alpha(\partial\beta),$$

and define it on generators as

$$\partial a = \sum_{\dim \mathcal{M}_A(a;\mathbf{b})=0} (-1)^{(n-1)(|a|+1)} \, |\mathcal{M}_A(a;\mathbf{b})| \; A\mathbf{b},$$

where $|\mathcal{M}_A(a;\mathbf{b})|$ is the algebraic number of rigid disks in the moduli space (where the sign of rigid disk is defined as in the previous section). It follows from [6] Lemma 1.5, that ∂ decreases grading by 1 in the connected case, see Remark 4.5 for the disconnected case. The purpose of the next subsection is to complete the proof of the following theorem.

Theorem 4.1. *The map $\partial \colon \mathcal{A}(L) \to \mathcal{A}(L)$ is a differential. That is,*

(4.2) $$\partial \circ \partial = 0.$$

Proof. It is a straightforward consequence of the signed Leibniz rule that the lemma follows once (4.2) has been established for generators. The fact that it holds for generators will be established below. $\square$

Remark 4.2. Note that the above definition also make sense for other coefficient rings. For example one could replace $\mathbb{Z}$ above with $\mathbb{Z}/k\mathbb{Z}$, for any $k \in \mathbb{Z}$ or by $\mathbb{Q}$.

4.2. **Orientations of 1-dimensional moduli spaces and their boundaries.** In order to prove Theorem 4.1, we will determine the relations between orientations on 1-dimensional moduli spaces and the signs of the pairs of rigid disks which are their boundaries.

4.2.1. *Even and odd punctures and grading.* We connect our abstract definitions of even and odd to the geometrical situation under study. Let c be a Reeb chord of an oriented connected chord generic Legendrian submanifold $L \subset \mathbb{C}^n \times \mathbb{R}$. Then the two tangent spaces of L, the projections of which intersect at $\Pi_{\mathbb{C}}(c) = c^*$, are oriented. We order these tangent spaces by taking the one with the *largest z-coordinate first*. The holomorphic disks we study have punctures mapping to Reeb chords. Translating the definition of even and odd punctures to the present situation, we say that a Reeb chord c is *even* if the even negative boundary condition $R_{ne}(1,\theta)$ takes the orientation of the upper tangent space to that of the lower. Otherwise we say that c is *odd* and, as is easy to see, in this case the odd negative boundary condition $R_{no}(1,\theta)$ takes the orientation of the upper tangent space to the orientation of the lower.

Lemma 4.3. *A Reeb chord c is even (odd) if and only if its grading $|c|$ is even (odd).*

Proof. Recall that

$$|c| = \mu(\gamma * \lambda) - 1,$$

where γ is a capping path of c and where λ is the J-trick path connecting the lower to the upper tangent space at c^*. Consider instead the inverse of the path $\gamma * \lambda$. This path is a negative rotation $\hat{\pi} - \theta$ where θ is the complex angle of the upper and lower tangent spaces followed by γ backwards (i.e. from bottom to top). The Maslov index of this path is even (odd) if and only if this path preserves (reverses) the orientation of the upper tangent plane.

Now, closing up with the negative even boundary condition path instead of the inverse J-trick path changes the Maslov index by one. Thus the negative even boundary condition path preserves orientation if and only if $|c|$ is even. $\qquad\square$

Remark 4.4. In the case of an oriented disconnected Legendrian submanifold we use this notion of even and odd punctures to define the $\mathbb{Z}_2$-grading discussed above.

Remark 4.5. To see that the differential decreases grading by 1 in the disconnected case note that (see Proposition 5.14 in [6]) the formal dimension of a component of a moduli space $\mathcal{M}(a; b_1, \ldots, b_k)$ with boundary mapping to the collection of paths $\lambda = (\lambda_1, \ldots, \lambda_{k+1})$ on the Legendrian submanifold $L \subset \mathbb{C}^n \times \mathbb{R}$ equals

$$n + \mu(\hat{\lambda}) + k - 2,$$

where $\mu(\hat{\lambda})$ is the Maslov index of the closed path $\hat{\lambda}$ obtained by closing up the paths λ at the corners by rotating the Lagrangian subspace of the incoming edge to the Lagrangian subspace at the outgoing edge in the negative direction. Note that at a negative corner the incoming Lagrangian subspace is the upper one. Thus, the negative close up preserves orientation if and only if the puncture is odd. At the positive puncture the incoming Lagrangian subspace is the lower one and the negative close up preserves the orientation if and only if n is even and the puncture is even or n is odd and the puncture is odd. Since the Maslov index of a loop of Lagrangian subspaces is even if and only if it is orientation preserving we find that

$$n + \mu(\hat{\lambda}) + k - 2 \equiv |a|_2 + \sum_{j=1}^{k} |b_j|_2 + 1 \mod 2,$$

where $|\cdot|_2$ denotes the modulo 2 grading. We conclude that the differential changes grading.

4.2.2. *Gluing conformal structures and orientations.* We shall determine the relation between orientations on spaces of conformal structures on a disk which is close to splitting up into two disks and the conformal structures on the two disks into which it splits. To this end we first describe the standard orientation on $\mathcal{C}_{m+1}$ in various local coordinates. Consider a disk D_{m+1} with $m + 1 > 3$ punctures $(p_0, \ldots, p_m)$ on the boundary representing the conformal structure κ and where p_0 is the positive puncture. Then the standard orientation of $\mathcal{C}_{m+1}$ at κ is given by the $(m-2)$-tuple of vectors $\partial_2, \ldots, \partial_m$ where ∂_j denotes the vector tangent to ∂D_{m+1} at p_j and points in the positive direction along ∂D_{m+1}. On the other hand we can coordinatize a neighborhood of κ in $\mathcal{C}_{m+1}$ by fixing any three punctures on ∂D_{m+1} and letting the remaining punctures move. We will however restrict attention to coordinates in which the positive puncture remains fixed. For such coordinates the positive orientation is given by the following lemma, the proof of which is straightforward.

Lemma 4.6. *The positive orientation of $\mathcal{C}_{m+1}$ at κ in coordinates obtained by fixing the ordered points (p_0, p_r, p_{r+s}) is given by the $(m-2)$-tuple of vectors*

$$\partial_1, \ldots, \partial_{r-1}, -\partial_{r+1}, -\partial_{r+2}, \ldots, -\partial_{r+s-2}, -\partial_{r+s-1}, \partial_{r+s+1}, \ldots, \partial_m.$$

Secondly, we look at the outward normal to the space of conformal structures at a disk which is obtained by gluing two disks. Let $m_j \geq 2$, $j = 1, 2$. Consider D_{m_1+1} and D_{m_2+1} with punctures $(q_0, \ldots, q_{m_1})$ and $(p_0, \ldots, p_{m_2})$, respectively. Let D_m be the disk obtained from gluing D_{m_1+1} to D_{m_2+1} by identifying q_0 and p_j, $1 \leq j \leq m_2$. To see the outward normal of the conformal structure of the glued disk we use coordinates on D_{m_2+1} which fixes p_0, p_j, and one of the punctures next to p_j, and coordinates on D_{m_1+1} which fixes q_0, q_1, and q_{m_1}. Note that D_m has punctures corresponding to all p_k except p_j and all q_k except q_0. We use coordinates on $T\mathcal{C}_m$ fixing p_0, q_1, and q_{m_1}. The outward normal is then represented by the tangent vector to the circle at the puncture formerly next to p_j directed away from

p_j. We next note that there are natural inclusions $T\mathcal{C}_{m_j+1} \to T\mathcal{C}_m$, $j = 1, 2$. These induce the decomposition $T\mathcal{C}_m = \mathbb{R} \oplus T\mathcal{C}_{m_1+1} \oplus T\mathcal{C}_{m_2+1}$, where the $\mathbb{R}$-direction is spanned by the normal direction discussed above. Let o_{m_j+1} be the standard orientation on $\mathcal{C}_{m_j+1}$, $j = 1, 2$ and let o be the orientation of the outward normal then

Lemma 4.7. *The direct sum map above induces the orientation* $-(-1)^{(m_1-1)j}$. *In other words,*

$$o \wedge o_{m_1+1} \wedge o_{m_2+1} = -(-1)^{(m_1-1)j} o_m.$$

Proof. We must consider two cases separately according to whether the fixed puncture in ∂D_{m_2+1} is p_{j-1} or p_{j+1}. Let ∂_{p_k} and ∂_{q_k} denote the positive tangent vector of the boundary of the glued disk at p_k and q_k, respectively. Assume first that p_{j-1} is fixed. Then by Lemma 4.6

$$o_m = \left(\partial_{p_1} \wedge \cdots \wedge \partial_{p_{j-2}}\right) \wedge \partial_{p_{j-1}} \wedge \left(-\partial_{q_2} \wedge \cdots \wedge -\partial_{q_{m_1-1}}\right) \wedge \left(\partial_{p_{j+1}} \wedge \cdots \wedge \partial_{p_{m_2}}\right)$$

$$= -(-1)^{(m_1-1)(j-2)} \left(-\partial_{p_{j-1}}\right) \wedge \left(-\partial_{q_2} \wedge \cdots \wedge -\partial_{q_{m_1-1}}\right)$$
$$\wedge \left(\partial_{p_1} \wedge \cdots \wedge \partial_{p_{j-2}}\right) \wedge \left(\partial_{p_{j+1}} \wedge \cdots \wedge \partial_{p_{m_2}}\right)$$

$$= -(-1)^{(m_1-1)j} o \wedge o_{m_1+1} \wedge o_{m_2+1}.$$

The computation for p_{j+1} fixed is similar. $\qquad\square$

4.2.3. *Disks with few punctures and marked points.* When proving the gluing theorem, Section 7 in [6], holomorphic disks with ≤ 2 punctures and holomorphic disks with ≥ 3 punctures were treated simultaneously by adding marked points to disks with few punctures. We will take the same approach here and thus we need to discuss orientations on determinants in the presence of marked points. Recall that we introduced marked points on a holomorphic disk $u \colon D_m \to \mathbb{C}^n$ with boundary on a Legendrian submanifold $L \subset \mathbb{C}^n \times \mathbb{R}$ by picking a codimension one submanifold H of L such that $u(p) \in H$ and such that $du(\partial_p)$ is transverse to H for some $p \in \partial D_m$. We then consider the $\bar\partial$-problem for maps $w \colon D_{m+1} \to \mathbb{C}^n$ in a neighborhood of u, which takes the additional marked point p into H and we use a neighborhood of u in that Sobolev space to find local coordinates on the moduli space. In our study of orientations below, we will take H as codimension one spheres in L around one of the endpoints of some Reeb chord to which some puncture of u maps. We start by describing the corresponding situation on the level of the linearized equation. Let $A \colon \partial D_m \to U(n)$ be a trivialized boundary condition for the $\bar\partial$-operator. Pick d points $q_1, \ldots, q_d$ in ∂D_m which are not punctures. Let $W_A = \mathcal{H}_2[A](D_m, \mathbb{C}^n)$ and let $V_A = \mathcal{H}_1[0]$ then $\bar\partial_A \colon W_A \to V_A$. For each q_j, pick a linear form $l_j \colon A(q_j)\mathbb{R}^n \to \mathbb{R}$ and consider the linear functionals

$$\alpha_j \colon W_A \to \mathbb{R}, \quad \alpha_j(w) = l_j(w(q_j)).$$

Let $W'_A = \bigcap_{j=1}^d \operatorname{Ker}(\alpha_j)$ and define the operator

$$\bar\partial'_A \colon W'_A \to V_A$$

as the restriction of $\bar\partial_A$. Then the index of $\bar\partial'_A$ satisfies

$$\operatorname{Index}(\bar\partial'_A) = \operatorname{Index} \bar\partial_A - d.$$

Pick d elements $\alpha^1, \ldots, \alpha^d \in W$ such that $\alpha_j(\alpha^k) = \delta_j^k$. This gives a direct sum decomposition

$$W_A = W'_A \oplus \mathbb{R}^d,$$

which induces the exact sequence

(4.3)
$$0 \longrightarrow \mathrm{Ker}(\bar{\partial}_A) \xrightarrow{\alpha} \mathrm{Ker}(\bar{\partial}'_A) \oplus \mathbb{R}^d \xrightarrow{\beta}$$
$$\mathrm{Coker}(\bar{\partial}'_A) \xrightarrow{\gamma} \mathrm{Coker}(\bar{\partial}_A) \longrightarrow 0,$$

where α is the direct sum decomposition followed by the L^2-projection to $\mathrm{Ker}(\bar{\partial}'_A)$ in the first summand, where β is $\bar{\partial}_A$ followed by projection to $\mathrm{Coker}(\bar{\partial}'_A)$, and where γ is the natural projection induced form the inclusion $\mathrm{Im}(\bar{\partial}'_A) \subset \mathrm{Im}(\bar{\partial}_A)$. This sequence induces an isomorphism

$$\Lambda^{\max} \mathrm{Ker}(\bar{\partial}_A) \otimes \Lambda^{\max} \mathrm{Coker}(\bar{\partial}_A)^* = \Lambda^{\max}(\mathrm{Ker}(\bar{\partial}'_A) \oplus \mathbb{R}^d) \otimes \Lambda^{\max} \mathrm{Coker}(\bar{\partial}'_A)^*.$$

To facilitate our sign discussions we will assume that d is *even*. In that case the position of $\mathbb{R}^d$ in the direct sum decomposition is of no importance for orientations and we have a canonical isomorphism

$$\Lambda^{\max}(\mathrm{Ker}(\bar{\partial}'_A) \oplus \mathbb{R}^d) = \Lambda^{\max} \mathrm{Ker}(\bar{\partial}'_A) \otimes \Lambda^{\max}(\mathbb{R}^d).$$

Thus, for d even, an orientation on $\mathbb{R}^d$ gives a canonical isomorphism between orientations on $\det(\bar{\partial}'_A)$ and orientations on $\det(\bar{\partial}_A)$. We next consider gluing isomorphisms in the presence of marked points. Let $A \colon D_m \to U(n)$ and $B \colon D_s \to U(n)$ be trivialized boundary conditions such that the positive puncture of A can be glued to some negative puncture of B. Assume that there are $2a$ marked points near the positive puncture on D_m and $2b$ marked points near the negative puncture of B to which A is glued. Consider the glued boundary condition $A \sharp B \colon D_{m+s-2} \to U(n)$ and note that D_{m+s-2} inherits the marked points from D_m and D_s, and thus comes equipped with $2(a + b)$ marked points. Consider the gluing sequence in Lemma 3.4. Note that the cut-off of a function in $\mathrm{Ker}(\bar{\partial}_A)$ ($\mathrm{Ker}(\bar{\partial}_B)$) vanishes on the part of D_{m+s-2} which correspond to D_s (D_m). This observation implies that the gluing sequence obtained by replacing the operators $\bar{\partial}_A$, $\bar{\partial}_B$, and $\bar{\partial}_{A\sharp B}$ in Lemma 3.4 with $\bar{\partial}'_A$, $\bar{\partial}'_B$, and $\bar{\partial}'_{A\sharp B}$, respectively, is also exact. Let o_A and o_B be orientations on $\det(\bar{\partial}_A)$ and $\det(\bar{\partial}_B)$. Orient $\mathbb{R}^{2a}$ and $\mathbb{R}^{2b}$. Let o'_A and o'_B be the induced orientations on $\det(\bar{\partial}'_A)$ and $\det(\bar{\partial}'_B)$ and endow $\mathbb{R}^{2a} \oplus \mathbb{R}^{2b}$ with the orientation induced from the orientations of its summands. Let $o_{A\sharp B}$ be the orientation induced on $\det(\bar{\partial}_{A\sharp B})$ from the gluing sequence of $\bar{\partial}_A$ and $\bar{\partial}_B$. Then $o_{A\sharp B}$ and the orientation on $\mathbb{R}^{2a} \oplus \mathbb{R}^{2b}$ induces an orientation $o'_{A\sharp B}$ on $\det(\bar{\partial}'_{A\sharp B})$.

Lemma 4.8. *The orientation on* $\det(\bar{\partial}'_{A\sharp B})$ *induced from the gluing sequence of* $\bar{\partial}'_A$ *and* $\bar{\partial}'_B$ *and the orientations* o'_A *and* o'_B *equals* $o'_{A\sharp B}$.

Proof. To show this we consider the following diagram.

$$
\begin{bmatrix} \mathrm{Ker}\,\bar{\partial}_A \\ \mathrm{Ker}\,\bar{\partial}_B \end{bmatrix} \longrightarrow \begin{bmatrix} \mathrm{Ker}\,\bar{\partial}'_A \\ \mathbb{R}^{2a} \\ \mathrm{Ker}\,\bar{\partial}'_B \\ \mathbb{R}^{2b} \end{bmatrix} \longrightarrow \begin{bmatrix} \mathrm{Coker}\,\bar{\partial}'_A \\ \mathrm{Coker}\,\bar{\partial}'_B \end{bmatrix} \longrightarrow \begin{bmatrix} \mathrm{Coker}\,\bar{\partial}_A \\ \mathrm{Coker}\,\bar{\partial}_B \end{bmatrix}
$$

$$
\uparrow \qquad\qquad \uparrow \qquad\qquad \downarrow \qquad\qquad \downarrow
$$

$$
\mathrm{Ker}\,\bar{\partial}_{A\sharp B} \longrightarrow \begin{bmatrix} \mathrm{Ker}\,\bar{\partial}'_{A\sharp B} \\ \mathbb{R}^{2a} \\ \mathbb{R}^{2b} \end{bmatrix} \longrightarrow \mathrm{Coker}\,\bar{\partial}'_{A\sharp B} \longrightarrow \mathrm{Coker}\,\bar{\partial}_{A\sharp B}
$$

$$
\uparrow \qquad\qquad \uparrow \qquad\qquad \downarrow \qquad\qquad \downarrow
$$

where the 0's on the left and right in the first and second horizontal rows have been dropped and where the 0's below the lower row of arrows have been dropped as well. The upper horizontal row is the direct sum of the sequences inducing o'_A and o'_B from o_A and o_B,

respectively. The lower horizontal row is the sequence inducing $o'_{A\sharp B}$ from $o_{A\sharp B}$. The gluing sequence for $\bar{\partial}_{A\sharp B}$ is obtained from the above diagram by adding an arrow from the top left entry to the top right entry. (Thinking of it as a half-circular arc the gluing sequence is then the outer boundary of the diagram.) The gluing sequence for $\bar{\partial}'_{A\sharp B}$ is obtained from the above diagram by adding an arrow from the middle left entry in top row to the middle right entry in the same row and forgetting the $\mathbb{R}^{2a}$ and $\mathbb{R}^{2b}$ summands. (Thinking also of this arrow as a half-circular arc the gluing sequence is the inner boundary of the diagram.) Let $(\sigma, \bar{\sigma})$ be wedges of vectors representing the orientation pair on the pair

$$\left(\mathrm{Ker}(\bar{\partial}_{A\sharp B}), \mathrm{Coker}(\bar{\partial}_{A\sharp B})\right).$$

The induced orientation on the pair

$$\left(\begin{bmatrix}\mathrm{Ker}(\bar{\partial}_A)\\ \mathrm{Ker}(\bar{\partial}_B)\end{bmatrix}, \begin{bmatrix}\mathrm{Coker}(\bar{\partial}_A)\\ \mathrm{Coker}(\bar{\partial}_B)\end{bmatrix}\right)$$

is then represented by

$$(\sigma \wedge \omega, \bar{\omega} \wedge \bar{\sigma}),$$

where ω is a wedge of vectors on the complement of the image of σ. This pair in turn induces the orientation pair

$$(\sigma \wedge \omega \wedge \phi, \bar{\phi} \wedge \bar{\omega} \wedge \bar{\sigma}),$$

on the pair

$$(4.4) \qquad \left(\begin{bmatrix}\mathrm{Ker}(\bar{\partial}'_A)\\ \mathbb{R}^{2a}\\ \mathrm{Ker}(\bar{\partial}'_B)\\ \mathbb{R}^{2b}\end{bmatrix}, \begin{bmatrix}\mathrm{Coker}(\bar{\partial}'_A)\\ \mathrm{Coker}(\bar{\partial}'_B)\end{bmatrix}\right)$$

where ϕ is a wedge of vectors on the complement of the image of $\sigma \wedge \omega$. On the other hand the orientation pair $(\sigma, \bar{\sigma})$ induces the orientation pair

$$(\sigma \wedge \theta, \bar{\theta} \wedge \bar{\sigma})$$

on the pair

$$\left(\begin{bmatrix}\mathrm{Ker}(\bar{\partial}'_{A\sharp B})\\ \mathbb{R}^{2a}\\ \mathbb{R}^{2b}\end{bmatrix}, \mathrm{Coker}(\bar{\partial}'_{A\sharp B})\right).$$

This in turn induces an orientation

$$(\sigma \wedge \theta \wedge \tau, \bar{\tau} \wedge \bar{\theta} \wedge \bar{\sigma})$$

on the pair in (4.4). Noting that the orientations of $\omega \wedge \phi$ and $\theta \wedge \tau$ agree if and only if those of $\bar{\omega} \wedge \bar{\phi}$ and $\bar{\theta} \wedge \bar{\tau}$ do and that the vertical maps on the $\mathbb{R}^{2a} \oplus \mathbb{R}^{2b}$ summand is the identity we find that the lemma holds. $\qquad\square$

We now turn to induced orientations on moduli spaces. We show that there is a natural way to pick marked points and an orientation on the corresponding copies of $\mathbb{R}$ so that the orientation of a moduli space of a disk with marked points is determined exactly as for disks with many punctures. In our applications we use only moduli spaces of dimensions 0 and 1 so we consider these two cases. We also assume we are in a generic situation where all moduli spaces are transversely cut out. Let $u\colon D_m \to \mathbb{C}^n$ be a holomorphic disk in a moduli space of dimension 0 or 1. Note then that the corresponding $\bar{\partial}_\kappa$-operator which is a part of the linearization of the $\bar{\partial}$-map at u (κ denotes the conformal structure) has either only kernel or only cokernel. Consider the case of only cokernel first. In that case we choose an even number of marked points $q_1, \ldots, q_d$ near a puncture of u. We obtain in that way a new source disk $\tilde{D}_{m,d}$ with m punctures and d marked points. Pick local 1-parameter families

of diffeomorphisms ϕ_j^τ around the marked points which move them in the positive direction along the boundary and set $\alpha_j(w) = \langle du(\partial_\tau \phi_j(q_j)), w(q_j) \rangle$. In this case the orientation inducing sequence reduces to

$$0 \longrightarrow \mathbb{R}^d \longrightarrow \operatorname{Coker}(\bar{\partial}'_\kappa) \longrightarrow \operatorname{Coker}(\bar{\partial}_\kappa) \longrightarrow 0.$$

Viewing $\operatorname{Coker}(\bar{\partial}_\kappa)$ as a subspace of $\mathcal{H}_{1,\epsilon}[0]$, we obtain a splitting $\operatorname{Coker}(\bar{\partial}_\kappa) \to \operatorname{Coker}(\bar{\partial}'_\kappa)$ and a corresponding direct sum decomposition $\operatorname{Coker}(\bar{\partial}'_\kappa) = \operatorname{Coker}(\bar{\partial}_\kappa) \oplus \mathbb{R}^d$. We obtain the diagram

$$
\begin{array}{ccc}
T\mathcal{C}_m \oplus \mathbb{R}^d & \longrightarrow & T\mathcal{C}_{m+d} \\
\downarrow & & \downarrow \\
\operatorname{Coker} \bar{\partial}_\kappa \oplus \mathbb{R}^d & \longrightarrow & \operatorname{Coker}(\bar{\partial}'_\kappa),
\end{array}
$$

where the top arrow is an isomorphism with inverse which maps the tangent vector of a moving marked point to the corresponding factor in $\mathbb{R}^d$. Note that the marked points in ∂D_m lies in an arc which contains no punctures. Thus, this map is orientation preserving since d is even. Likewise the leftmost vertical map is the identity on the $\mathbb{R}^d$-component. It follows that the orientation on the moduli space induced by $T\mathcal{C}_m \to \operatorname{Coker}(\bar{\partial}_\kappa)$ is the same as the one induced by $T\mathcal{C}_{m+d} \to \operatorname{Coker}(\bar{\partial}'_\kappa)$. Consider next the case when $\operatorname{Coker}(\bar{\partial}) = 0$. In this case, if d is sufficiently large then $\operatorname{Ker}(\bar{\partial}') = 0$ and the orientation inducing sequence reduces to

$$0 \longrightarrow \operatorname{Ker}(\bar{\partial}) \longrightarrow \mathbb{R}^d \longrightarrow \operatorname{Coker}(\bar{\partial}') \longrightarrow 0.$$

Thus we get $\operatorname{Coker}(\bar{\partial}') = \mathbb{R}^d / \operatorname{Ker}(\bar{\partial})$. We use the convention that marked points are added near the positive puncture in the negative direction of it. We then get the diagram

$$
\begin{array}{ccc}
\mathbb{R}^d/T\mathcal{A}_m & \longrightarrow & T\mathcal{C}_{3-m+d} \\
\downarrow & & \downarrow \\
\mathbb{R}^d/\operatorname{Ker}(\bar{\partial}) & \longrightarrow & \operatorname{Coker}(\bar{\partial}')
\end{array}
,
$$

where the top horizontal map is the quotient of the (oriented) automorphism group acting on the positively oriented tangent vectors to the marked points moving on the boundary which is then orientation preserving, and where the left vertical arrow is the map induced by $\mathcal{A}_m \to \operatorname{Ker}(\bar{\partial})$. It follows again that the orientation on the moduli space induced by $T\mathcal{A}_m \to \operatorname{Ker}(\bar{\partial})$ is the same as that induced by $T\mathcal{C}_{3-m+d} \to \operatorname{Coker}(\bar{\partial}')$.

4.2.4. *Gluing linearized operators corresponding to rigid disks.* We now determine how capping orientations behave under gluing as in Lemma 3.4. In fact, we concentrate on the case important for our applications: when two *rigid* disks u and v with boundary on a Legendrian submanifold $L \subset \mathbb{R} \times \mathbb{C}^n$ are being glued. As we have seen gluings of disks with few punctures (when kernels of $\bar{\partial}$-operators are present) can be treated as gluings of disks with many punctures. We concentrate on the case of many punctures first and explain in Remark 4.10 how to modify the arguments in the presence of marked points. Let A be a trivialized stabilized boundary condition on the m-punctured disk (corresponding to u above), with positive puncture at a Reeb chord b_k and negative punctures at Reeb chords $f_1, \ldots, f_{m-1}$. Let B be a trivialized stabilized boundary condition on the $(r+1)$-punctured disk (corresponding to v above), with positive puncture at a Reeb chord a and negative punctures at Reeb chords $b_1, \ldots, b_k, \ldots, b_r$. Let $\bar{\partial}_{A\sharp B}$ be the operator which is obtained by gluing $\bar{\partial}_A$ and $\bar{\partial}_B$ at b_k. Since we model the case when the disks u and v are rigid and have many punctures we assume that $\operatorname{Ker}(\bar{\partial}_A) = 0 = \operatorname{Ker}(\bar{\partial}_B)$ and that

$$\operatorname{Index}(\bar{\partial}_A) = -(m-3), \qquad \operatorname{Index}(\bar{\partial}_B) = -(r-2).$$

In particular, it follows from the dimension formula that

$$(4.5) \qquad |b_k| - \sum_{j=1}^{m-1} |f_j| = 1 - |A_u|$$

$$(4.6) \qquad |a| - \sum_{j=1}^{r} |b_j| = 1 - |B_v|,$$

where $A_u, B_v \in H_1(L)$ are the homology classes of the boundary paths of u and v capped off with the capping paths at each puncture. Since L is orientable, $|A_u|$ and $|B_v|$ are even. Let o_A, o_B and $o_{A\sharp B}$ denote the capping orientations of the determinant lines of $\bar\partial_A$, $\bar\partial_B$, and $\bar\partial_{A\sharp B}$.

Lemma 4.9. *Let $\sigma_j = \pm 1$, $j = 1, 2$. The gluing map in Lemma 3.4 induces from orientations $\sigma_1 o_A$ and $\sigma_2 o_B$ the orientation*

$$(-1)^{n+1}(-1)^{mk}(-1)^{\sum_{j=1}^{k-1}|b_j|}(-1)^{(|b_k|+1)(n-1)}(\sigma_1\sigma_2)o_{A\sharp B}.$$

Proof. Consider the boundary condition $\hat A \sharp \hat B$ obtained by gluing the two boundary conditions $\hat A$ and $\hat B$. This boundary condition can be obtained in two ways.

  (i) Glue all capping disks to A and B, respectively and then join $\hat A$ and $\hat B$.
  (ii) Glue the operators $\bar\partial_{b_k+}$ and $\bar\partial_{b_k-}$ to obtain an operator $\bar\partial_E$ on the closed disk. Glue all relevant capping disks to $A\sharp B$ giving the boundary condition $\widehat{A\sharp B}$ on the closed disk. Then glue $\bar\partial_E$ to $\bar\partial_{\widehat{A\sharp B}}$.

Repeated application of Lemmas 3.5 and 3.6 gives the following gluing sequence corresponding to the gluing (i):

$$0 \longrightarrow \mathrm{Ker}(\bar\partial_{\hat A \sharp \hat B}) \longrightarrow \begin{bmatrix} \bigoplus_{j=1}^{m-1} \mathrm{Ker}(\bar\partial_{f_j-}) \\ \bigoplus_{j=1}^{r} \mathrm{Ker}(\bar\partial_{b_j-}) \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} \bigoplus_{j=1}^{m-1} \mathrm{Coker}(\bar\partial_{f_j-}) \\ \mathrm{Coker}(\bar\partial_{b_k+}) \\ \mathrm{Coker}(\bar\partial_A) \\ \mathbb{R}^n \\ \bigoplus_{j=1}^{r} \mathrm{Coker}(\bar\partial_{b_j-}) \\ \mathrm{Coker}(\bar\partial_{a+}) \\ \mathrm{Coker}(\bar\partial_B) \end{bmatrix} \longrightarrow \mathrm{Coker}(\bar\partial_{\hat A \sharp \hat B}) \longrightarrow 0.$$

Note that the capping orientation o_A together with the orientations on the capping disks of A induce the canonical orientation on $\det(\bar\partial_{\hat A})$, by definition. Similarly, the capping orientation o_B together with the orientations on the capping disks of B induce the canonical orientation on $\det(\bar\partial_{\hat B})$. Thus Lemma 3.11 implies that that these orientations in the above sequence induce the canonical orientation on $\bar\partial_{\hat A \sharp \hat B}$.

To find the corresponding gluing sequence for (ii) we first look at the gluing sequence for $\bar\partial_{A\sharp B}$. This sequence is the following.

$$0 \longrightarrow \mathrm{Coker}(\bar\partial_A) \oplus \mathrm{Coker}(\bar\partial_B) \longrightarrow \mathrm{Coker}(\bar\partial_{A\sharp B}) \longrightarrow 0,$$

It allows us to identify the two spaces involved. Second the gluing sequence for $\bar\partial_E$ is

$$0 \longrightarrow \mathrm{Ker}(\bar\partial_E) \longrightarrow \mathrm{Ker}(\bar\partial_{b_k-}) \longrightarrow$$

$$\begin{bmatrix} \mathrm{Coker}(\bar\partial_{b_k+}) \\ \mathrm{Coker}(\bar\partial_{b_k-}) \end{bmatrix} \longrightarrow \mathrm{Coker}(\bar\partial_E) \longrightarrow 0,$$

where both the leftmost and the rightmost non-trivial maps are isomorphisms. Moreover, by definition, the chosen orientation pair on

$$(\mathrm{Ker}(\bar{\partial}_{b_k-}), \mathrm{Coker}(\bar{\partial}_{b_k+}) \oplus \mathrm{Coker}(\bar{\partial}_{b_k-}))$$

corresponds to the canonical orientation on $\bar{\partial}_E$. With these two identifications made we apply Lemmas 3.5 and 3.6 and find that the gluing sequence in case (ii) is

$$0 \longrightarrow \mathrm{Ker}(\bar{\partial}_{\hat{A}\sharp\hat{B}}) \longrightarrow \begin{bmatrix} \mathrm{Ker}(\bar{\partial}b_k-) \\ \bigoplus_{j=1}^{k-1} \mathrm{Ker}(\bar{\partial}_{b_j-}) \\ \bigoplus_{j=1}^{m-1} \mathrm{Ker}(\bar{\partial}_{\underline{f}_j-}) \\ \bigoplus_{j=k+1}^{r} \mathrm{Ker}(\bar{\partial}_{b_j-}) \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} \mathrm{Coker}(\bar{\partial}_{b_k+}) \\ \mathrm{Coker}(\bar{\partial}_{b_k-}) \\ \mathbb{R}^n \\ \bigoplus_{j=1}^{k-1} \mathrm{Coker}(\bar{\partial}_{b_j-}) \\ \bigoplus_{j=1}^{m-1} \mathrm{Coker}(\bar{\partial}_{\underline{f}_j-}) \\ \bigoplus_{j=k+1}^{r} \mathrm{Coker}(\bar{\partial}_{b_j-}) \\ \mathrm{Coker}(\bar{\partial}_{a+}) \\ \mathrm{Coker}(\bar{\partial}_A) \\ \mathrm{Coker}(\bar{\partial}_B) \end{bmatrix} \longrightarrow \mathrm{Ker}(\bar{\partial}_{\hat{A}\sharp\hat{B}}) \longrightarrow 0.$$

Lemma 3.11 implies that the chosen orientations on all capping disks together with the capping orientation $o_{A\sharp B}$ on $\mathrm{Coker}(\bar{\partial}_{A\sharp B}) = \mathrm{Coker}(\bar{\partial}_A) \oplus \mathrm{Coker}(\bar{\partial}_B)$ induce the canonical orientation on $\bar{\partial}_{\hat{A}\sharp\hat{B}}$. We are now in position to determine the gluing sign. Note first that in both sequences the middle map takes the complement of $\mathrm{Ker}(\bar{\partial}_{b_k-})$ to 0 and maps this vector non-trivially into the $\mathbb{R}^n$ summand in the next term in the sequence. To compare the signs we first move $\mathbb{R}^n$ so that it becomes the first summand in its respective sum in both sequences. Note that because of (4.5), the dimension of the space

$$\begin{bmatrix} \bigoplus_{j=1}^{m-1} \mathrm{Coker}(\bar{\partial}_{\underline{f}_j-}) \\ \mathrm{Coker}(\bar{\partial}_{b_k+}) \end{bmatrix}$$

equals $(n-2)$ + even if b_k is odd and $(n-1)$ + odd if b_k is even. Thus, the dimension of this space is congruent to n modulo 2 in either case. In sequence (i), moving $\mathbb{R}^n$ to the first position thus gives an orientation change of sign

$$(-1)^{n(m-3)}(-1)^{n^2} = (-1)^{n(m-2)}.$$

To find the corresponding change in the sequence (ii) note that the dimension of the space

$$\begin{bmatrix} \mathrm{Coker}(\bar{\partial}_{b_k+}) \\ \mathrm{Coker}(\bar{\partial}_{b_k-}) \end{bmatrix}$$

equals $n - 1$ and we find that the orientation change is

$$(-1)^{n(n-1)} = 1.$$

The orientation difference is thus

$$(-1)^{n(m-2)}.$$

With $\mathbb{R}^n$ moved to the first position we compute the sign change by changing the order of summands in sequence (i) to agree with those in sequence (ii): The orientation difference in the two sequences arising from their second terms are

$$(-1)^{m-1+k-1} \cdot (-1)^{(m-1)(k-1)} = (-1)^{mk-1},$$

where the first factor comes from moving $\mathrm{Ker}(\bar{\partial}_{b_k-})$ to the left and the second from moving $\oplus_j \mathrm{Ker}(\bar{\partial}_{f_j-})$ to its position. The orientation difference arising in the third term in the sequences (with $\mathbb{R}^n$ already moved to the first position) is calculated as follows. First we move the $\mathrm{Coker}(\bar{\partial}_{b_k+})$-term in sequence (i) to the left. This subspace has dimension $n-1$ if $|b_k|$ is even and $n-2$ if $|b_k|$ is odd. Since $|b_k| - \sum_j |f_j| = 1$ modulo 2, by (4.5), we see that if $|b_k|$ is odd then there is no sign arising from this move since $\dim(\bigoplus_j \mathrm{Coker}(\bar{\partial}_{f_j}))$ is even. On the other hand if $|b_k|$ is even we find the sign $(-1)^{(n-1)}$. In general the sign is thus

$$(-1)^{(|b_k|+1)(n-1)}.$$

Second we move the subspace $\bigoplus_j \mathrm{Coker}(\bar{\partial}_{b_j-}) \oplus \mathrm{Coker}(\bar{\partial}_{a+})$ over $\mathrm{Coker}(\bar{\partial}_A)$. The latter space has dimension $m-3$. We must calculate the dimension modulo 2 of the former. Since $|a| - \sum_j |b_j| = 1$ modulo 2, we find that the dimension of this space is $(n-1+\text{odd})$ if a is even and $(n-2+\text{even})$ if a is odd. Thus in any case we get a sign

$$(-1)^{(m-3)(n-2)}$$

from this motion. Finally we must move out $\mathrm{Coker}(\bar{\partial}_{b_k-})$ if this term is non-zero and move $\bigoplus_j \mathrm{Coker}(\bar{\partial}_{f_j-})$ to the right position. We consider two cases. First if b_k is even then $\mathrm{Coker}(\bar{\partial}_{b_k-})$ is zero dimensional and $\bigoplus_j \mathrm{Coker}(\bar{\partial}_{f_j-})$ is odd dimensional giving a sign $(-1)^{\sum_{j=1}^{k-1} |b_j|}$. On the other hand if b_k is odd then $\bigoplus_j \mathrm{Coker}(\bar{\partial}_{f_j-})$ is even dimensional and the sign is still $(-1)^{\sum_{j=1}^{k-1} |b_j|}$ since we must move the 1-dimensional space $\mathrm{Coker}(\bar{\partial}_{b_k-})$. Thus in any case we get

$$(-1)^{\sum_{j=1}^{k-1} |b_j|}.$$

Multiplying out all the signs gives the over all sign

$$- (-1)^{(|b_k|+1)(n-1)}(-1)^{mk}(-1)^{(\sum_{j=1}^{k-1} |b_j|)}(-1)^{(n-2)(m-3)}(-1)^{n(m-2)}$$

$$= (-1)^{n+1}(-1)^{mk}(-1)^{\sum_{j=1}^{k-1} |b_j|}(-1)^{(|b_k|+1)(n-1)},$$

as claimed. □

Remark 4.10. In the presence of marked points we simply need to replace the operators $\bar{\partial}_A$, $\bar{\partial}_B$, and $\bar{\partial}_{A\sharp B}$ above with $\bar{\partial}'_A$, $\bar{\partial}'_B$, and $\bar{\partial}'_{A\sharp B}$, respectively. In particular by adding sufficiently large even numbers of marked points we can assure that none of the latter operators has non-trivial kernel. Of course adding marked points effects the dimensions of the cokernels of the operators but the final expression for the sign is independent of these dimensions so the lemma holds in the presence of marked points as well.

4.2.5. *Orientations of* 1-*dimensional moduli spaces.* Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be a generic Legendrian submanifold oriented and with a fixed spin structure. Then as shown above, all moduli spaces of holomorphic disks with boundary on L come equipped with orientations. We show the following result. Let $\mathcal{M}$ be a component of a 1-dimensional moduli space with boundary $\partial \mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1$. For $\mathcal{M}_j$, $j = 0, 1$, let $\sigma(\mathcal{M}_j) = 1$ if the orientation of $\mathcal{M}$ is outwards at $\mathcal{M}_j$ and let $\sigma(\mathcal{M}_j) = -1$ otherwise. A boundary component $\mathcal{M}_j$ is a broken holomorphic disk. That is, two rigid holomorphic disks u and v such that the positive puncture of u is identified with some negative puncture of v. Assume that the positive puncture of u is b_k and that its negative punctures are $f_1, \ldots, f_{m-1}$. Assume also that the positive puncture of v is a and that its negative punctures are $b_1, \ldots, b_k, \ldots, b_r$.

Lemma 4.11. *Let μ_1 and μ_2 be the signs of u and v, respectively. Then*

$$\sigma(\mathcal{M}_j) = \mu_1 \mu_2 (-1)^n (-1)^{\sum_{j=1}^{k-1} |b_j|} (-1)^{(n-1)(|b_k|+1)}.$$

Proof. It is enough to consider the case of many punctures, see Remark 4.10, so assume $m \geq 3$ and $r \geq 2$. Let A and B be the boundary conditions corresponding to u and v, respectively. Consider the commutative diagram

$$
\begin{array}{ccc}
\mathbb{R} \oplus T\mathcal{C}_m \oplus T\mathcal{C}_{r+1} & \longrightarrow & T\mathcal{C}_{m+r-1} \\
\downarrow & & \downarrow \\
\mathrm{Coker}(\bar{\partial}_A) \oplus \mathrm{Coker}(\bar{\partial}_B) & \longrightarrow & \mathrm{Coker}(\bar{\partial}_{A \sharp B}).
\end{array}
$$

Here $T\mathcal{C}_m$, $T\mathcal{C}_{r+1}$, and $T\mathcal{C}_{m+r-1}$ are endowed with their standard orientations and $\mathbb{R}$ is endowed with the orientation corresponding to the *outward* normal ν. Moreover, the spaces appearing in the lower horizontal row are endowed with their capping orientations. We compute the orientations. The leftmost vertical map restricted to the complement of $\mathbb{R}$ is an isomorphism between oriented spaces of sign $\mu_1 \mu_2$. The lower horizontal arrow is an isomorphism of oriented vector spaces with sign

$$
\sigma = (-1)^{n+1}(-1)^{mk}(-1)^{\sum_{j=1}^{k-1} |b_j|}(-1)^{(n-1)(|b_k|+1)}
$$

by Lemma 4.9. The upper horizontal arrow is an isomorphism of sign $\tau = -(-1)^{mk}$ by Lemma 4.7. Now, the oriented tangent space to $\mathcal{M}$ is identified with the oriented kernel of the rightmost vertical map. Chasing the diagram we find that this oriented kernel is identified with $\mathbb{R}$ oriented by

$$
\tau \sigma \mu_1 \mu_2 \cdot \nu = (-1)^n (-1)^{\sum_{j=1}^{k-1} |b_j|}(-1)^{(n-1)(|b_k|+1)} \mu_1 \mu_2 \cdot \nu,
$$

as claimed. $\qquad\square$

Finally we are in position to complete the proof of Theorem 4.1.

Completion of the proof of Theorem 4.1. Let a be a Reeb chord of L and let $Cb_1, \ldots, b_r$ be a word appearing in $\partial \partial a$. Consider a component of a 1-dimensional moduli space

$$
\mathcal{M} = \mathcal{M}_C(a; b_1, \ldots, b_r),
$$

with oriented boundary $\partial \mathcal{M} = \mathcal{M}_1 - \mathcal{M}_0$. Assume that $\mathcal{M}_0$ consists of two broken disks in

$$
(4.7) \qquad \mathcal{M}_A(a; b_1, \ldots, b_{k-1}, f, b_{k+s}, \ldots, b_r) \text{ and } \mathcal{M}_B(f; b_k, \ldots, b_{k+s-1}),
$$

respectively, where $A + B = C$, and that $\mathcal{M}_1$ consists of two broken disks in

$$
(4.8) \qquad \mathcal{M}_{A'}(a; b_1, \ldots, b_{k'-1}, f', b_{k'+s'}, \ldots, b_r) \text{ and } \mathcal{M}_{B'}(f'; b_{k'}, \ldots, b_{k'+s'-1}),
$$

respectively, where $A' + B' = C$. By the definition of the differential the disks in (4.7) contribute

$$
(-1)^{(n-1)(|a|+1)}(-1)^{|A|}(-1)^{\sum_{j=1}^{k-1} |b_j|}(-1)^{(n-1)(|f|+1)} \mu_1 \mu_2 (A+B) b_1 \ldots b_r =
$$
$$
(-1)^{(n-1)(|a|+1)}(-1)^{\sum_{j=1}^{k-1} |b_j|}(-1)^{(n-1)(|f|+1)} \mu_1 \mu_2 C b_1 \ldots b_r,
$$

where μ_1 and μ_2 are the signs of the two disks and where we use the fact that $|A|$ is even. On the other hand the disks in (4.8) contribute

$$
(-1)^{(n-1)(|a|+1)}(-1)^{|A'|}(-1)^{\sum_{j=1}^{k'-1} |b'_j|}(-1)^{(n-1)(|f'|+1)} \mu'_1 \mu'_2 (A'+B') b_1 \ldots b_r =
$$
$$
(-1)^{(n-1)(|a|+1)}(-1)^{\sum_{j=1}^{k'-1} |b_j|}(-1)^{(n-1)(|f'|+1)} \mu'_1 \mu'_2 C b_1 \ldots b_r,
$$

where μ_1' and μ_2' are the signs of the two disks and where we use the fact that $|A'|$ is even. By Lemma 4.11

$$-(-1)^n = (-1)^n \sigma(\mathcal{M}_0) = (-1)^{\sum_{j=1}^{k-1} |b_j|} (-1)^{(n-1)(|f|+1)} \mu_1 \mu_2,$$

$$(-1)^n = (-1)^n \sigma(\mathcal{M}_1) = (-1)^{\sum_{j=1}^{k'-1} |b_j|} (-1)^{(n-1)(|f'|+1)} \mu_1' \mu_2'.$$

Thus these two terms cancel in $\partial\partial a$. Since all terms contributing to $\partial\partial a$ arise in this way we conclude $\partial\partial a = 0$. $\qquad\square$

4.3. **Stable tame isomorphism invariance of $\mathcal{A}(L)$.** We show that Legendrian isotopies of a Legendrian submanifold $L \subset \mathbb{C}^n \times \mathbb{R}$ with a fixed spin structure does not change the stable tame isomorphism class of its associated algebra $\mathcal{A}(L)$. More precisely

Theorem 4.12. *Let $L_t \subset \mathbb{C}^n \times \mathbb{R}$, $0 \leq t \leq 1$ be a Legendrian isotopy then the DGA $(\mathcal{A}(L_0), \partial_0)$ of L_0 and the DGA $(\mathcal{A}(L_1), \partial_1)$ of L_1 are stable tame isomorphic. (In particular the contact homology of L_0 is isomorphic to that of L_1.)*

Proof. The theorem follows from Lemma 4.13 and Corollaries 4.22 and Lemma 4.28 which are proved below. $\qquad\square$

Our approach is to study the bifurcations in moduli spaces under generic isotopies. During such deformations three events effect the moduli-spaces (and therefore possibly the differential): they undergo Morse modifications, there appear disks of formal dimension -1 (so called handle slide disks because of analogous phenomena in Morse theory), and there appear self tangency instances. To prove invariance it is sufficient to study these events separately. To show invariance in the first case we use a parameterized moduli space. To show invariance in the second we use an auxiliary Legendrian embedding of $L \times [0,1]$ with certain boundary conditions at $L \times \{0\}$ and $L \times \{1\}$, and its differential graded algebra. We mention these proofs have an advantage over the more straight forward proofs of invariance given in [6]. In particular, we avoid the need for "degenerate gluing", which is technically much more difficult than the gluing results needed to prove $\partial^2 = 0$.

4.3.1. *Invariance under trivial isotopies.* Let $L_t \subset \mathbb{C}^n \times \mathbb{R}$, be a generic 1-parameter family of Legendrian submanifolds such that L_0 and L_1 are generic and such that there are neither handle slide moments nor self tangency moments during the isotopy. that there are neither handle slide moments nor self tangency moments during the isotopy.

Lemma 4.13.

$$(\mathcal{A}(L_0), \partial_0) = (\mathcal{A}(L_1), \partial_1).$$

Proof. Note that since the complex angles in the added directions are taken to be smaller than any complex angle of a double point of L_t, the capping orientations are continuous in t. Therefore, the parameterized moduli spaces are naturally oriented. In particular, the rigid disks on L_0 and L_1 equals the oriented boundary of the one-dimensional parameterized moduli space. This oriented cobordism immediately gives $\partial_1 = \partial_0$. $\qquad\square$

4.3.2. *An auxiliary Legendrian submanifold.* We associate to a 1-parameter family of Legendrian embeddings $\phi_t \colon L \to \mathbb{C}^n \times \mathbb{R}$, $0 \leq t \leq 1$, Legendrian embeddings $\Phi_f^\delta \colon L \times \mathbb{R} \to \mathbb{C}^{n+1} \times \mathbb{R}$, depending on $\delta > 0$ and a positive Morse function $f \colon \mathbb{R} \to \mathbb{R}$. Let $\phi_t \colon L \subset \mathbb{C}^n \times \mathbb{R}$, $t \in [-1, 1]$ be a Legendrian isotopy. For small $\delta > 0$, fix smooth non-decreasing functions

(4.9) $$\alpha^\delta \colon [-1, 1] \to [-\delta, \delta]$$

such that $\alpha^\delta(\pm t) = \pm\delta$ for $\frac{3}{4} \leq t \leq 1$, and such that $\alpha^\delta(t) = \delta t$ for $-\frac{1}{4} \leq t \leq \frac{1}{4}$. Note that $\alpha^\delta \to 0$ as $\delta \to 0$. Fix standard coordinates

$$\big((x_1, y_1, \ldots, x_n, y_n), z\big) = (x, y, z)$$

on $\mathbb{C}^n \times \mathbb{R}$. Define ϕ_t^δ, $t \in \mathbb{R}$ as

$$\phi_t^\delta = \begin{cases} \phi_{-\delta} & \text{for } t \in (-\infty, -1], \\ \phi_{\alpha^\delta(t)} & \text{for } t \in [-1, 1], \\ \phi_\delta & \text{for } t \in [1, \infty). \end{cases}$$

Write

$$\phi_t^\delta(q) = \big(x_t(q), y_t(q), z_t(q)\big), \quad q \in L.$$

Fix a *positive* Morse function $f \colon \mathbb{R} \to \mathbb{R}$ and $\delta > 0$. Let $f'(t) = \frac{df}{dt}$ denote the derivative of f. Define $\Phi_f^\delta \colon \mathbb{R} \times L \to \mathbb{C} \times \mathbb{C}^n \times \mathbb{R}$,

$$\Phi_f^\delta(t, q) = \big(x_0(t, q), y_0(t, q), x(t, q), y(t, q), z(t, q)\big), \quad (t, q) \in \mathbb{R} \times L,$$

where

$$x_0(q, t) = t,$$
$$y_0(q, t) = f(t)\left(\frac{\partial z_t}{\partial t} - y_{j_t}(q)\frac{\partial x_{j_t}}{\partial t}\right) + f'(t)z_t(q),$$
$$x(q, t) = x_t(q),$$
$$y(q, t) = f(t)y_t(q),$$
$$z(q, t) = f(t)z_t(q).$$

It is straightforward to check that Φ_f^δ is a Legendrian embedding. Assume that the Morse function $f \colon \mathbb{R} \to \mathbb{R}$ above has local minima at ± 1 and no critical points in the region $(-\infty, -1) \cup (1, \infty)$. Then the x_0-coordinate c_0 of each Reeb chord c of Φ satisfies $|c_0| \leq 1$. Let u be a holomorphic disk with boundary on Φ_f^δ and one positive puncture.

Lemma 4.14. *If the positive puncture of u maps to a Reeb chord c of Φ with $c_0 = \pm 1$. Then the image of u lies in $\{x_0 = \pm 1\}$.*

Proof. For definiteness assume the positive puncture of u maps to c with $c_0 = 1$. Project Φ_f^δ to the (x_0, y_0)-plane. The image of this projection is contained in the region

$$\big\{-\alpha|x_0 - 1| \leq y_0 \leq \alpha|x_0 - 1|\big\}$$

for some α. If the projection u_0 of u to the (x_0, y_0)-plane is non-constant then it covers at least one of the regions

$$\big\{-\alpha|x_0 - 1| > y_0\big\} \cap B_r\big((1, 0)\big) \quad \text{or} \quad \big\{\alpha|x_0 - 1| < y_0\big\} \cap B_r\big((1, 0)\big),$$

for some ball $B_r\big((1, 0)\big)$. Since u has boundary on Φ_k^δ, u_0 takes no boundary point to the line $\{x_0 = 1\}$. This and the above covering property contradicts u_0 being bounded in the y_0-direction. The lemma follows. $\square$

Lemma 4.15. *The image of every holomorphic disk with boundary on Φ_f^δ is contained in the region $\{|x_0| \leq 1\}$.*

Proof. Arguing as in the proof of Lemma 4.14 we find that the projection to the (x_0, y_0)-plane of a holomorphic disk with boundary on Φ_f^δ can not intersect the lines $\{x_0 = \pm 1\}$ in interior points. It follows that the image of any disk lies entirely in one of the regions $\{x_0 \leq -1\}$, $\{|x_0| \leq 1\}$, or $\{x_0 \geq 1\}$. However, a disk with image in $\{x_0 \geq 1\}$ ($\{x_0 \leq -1\}$) must have

its positive puncture at a Reeb chord c with $c_0 = 1$ ($c_0 = -1$). The lemma follows from Lemma 4.14. □

Assume now that $\phi_\delta(L)$ is generic for each $\delta \neq 0$. Then it is a consequence of Lemma 6.25 in [6] that any rigid disk with boundary on Φ_f^δ and positive corner at some Reeb chord c with $c_0 = \pm 1$ is transversely cut out. Moreover by Lemma 6.12 in [6] transversality of the $\bar{\partial}$-equation can be achieved by perturbation near the positive puncture of a disk and it follows that there exists (arbitrarily small) perturbations of Φ_f^δ which are supported in the region $\{|x_0| < 1\}$ and which makes every moduli space (of formal dimension ≤ 1) transversely cut out. We fix such a perturbation of Φ_f^δ but keep the notation Φ_f^δ for the perturbed Legendrian embedding. Let $\mathcal{A}(\Phi_f^\delta)$ denote the algebra over $\mathbb{Z}[H_1(\mathbb{R} \times L)] = \mathbb{Z}[H_1(L)]$ generated by the Reeb chords of Φ_f^δ as in Subsection 4.1 and define the map (differential) ∂ of $\mathcal{A}(\Phi_f^\delta)$ as there.

Lemma 4.16. *The map $\partial\colon \mathcal{A}(\Phi_f^\delta) \to \mathcal{A}(\Phi_f^\delta)$ satisfies $\partial \circ \partial = 0$.*

Proof. In the light of Lemma 4.15, a word by word repetition of the proof of Theorem 4.1 establishes the lemma. □

4.3.3. *Invariance under handle slides.* Let $\phi_t\colon L \to \mathbb{C}^n \times \mathbb{R}$, $-1 \leq t \leq 1$ be a Legendrian isotopy such that L_0 is a generic handle slide moment. That is, there exists one handle slide disk in some $\mathcal{M}_A(a; \mathbf{b})$, which is the only non-empty moduli space of formal negative dimension, that all moduli spaces of holomorphic disk with boundary on $\phi_t(L) = L_t$, $t \neq 0$ of negative formal dimension are empty, and that all moduli spaces of rigid disks are transversally cut out. We choose notation so that $\{b_1, \ldots, b_r, a, c_1, \ldots, c_s\}$ are the Reeb chords of L_0 and so that

$$\mathcal{Z}(b_1) \leq \cdots \leq \mathcal{Z}(b_r) \leq \mathcal{Z}(a) \leq \mathcal{Z}(c_1) \leq \cdots \leq \mathcal{Z}(c_s).$$

Recall for a Reeb chord c, $\mathcal{Z}(c)$ is the difference in z-coordinates of its endpoints in $\mathbb{R}^{2n+1}$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a positive Morse function with local minima at ± 1, no critical points in the region $(-\infty, -1) \cup (1, \infty)$, and one local maximum at 0.

Lemma 4.17. *For all sufficiently small $\delta > 0$ the Reeb chords of Φ_f^δ are*

$$\left\{b_j[-1], b_j[1], b_j[0]\right\}_{j=1}^r \cup \left\{a[-1], a[1], a[0]\right\} \cup \left\{c_j[-1], c_j[1], c_j[0]\right\}_{j=1}^s,$$

where for any Reeb chord c of L_0, $|c[-1]| = |c[1]| = |c[0]| - 1 = |c|$.

Proof. It is easy to see that for $\delta = 0$ the Reeb chords are as described above and that the corresponding double points in $\mathbb{C} \times \mathbb{C}^n$ are transverse. This shows that the Reeb chords are as claimed for all sufficiently small δ. The second statement in the lemma is a straightforward consequence of the following grading formula from [5]. Let $c^\pm$ be the z-coordinates of the upper and lower points in the front projection corresponding to the Reeb chord c. Assume c^+ is above c^-. Near c^+ we can represent the front as the graph of a function $h_+ : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ with $h_+(q) = c^+$ for some $q \in U$. We can similarly find a function h_- for the front near c^-. Let $h = h_+ - h_-$. Since $c^\pm$ correspond to a double point in the Lagrangian projection, h has a critical point at q. If c is a transverse double point q is a non-degenerate critical point. From [5] we have

(4.10) $|c| = D(\gamma) - U(\gamma) + \text{Morse Index}(d^2h) - 1,$

where γ is a path in the front connecting c^+ to c^- and $D(\gamma)$ and $U(\gamma)$ is the number of down- and up-cusps of γ. □

We call $b_j[0]$, $a[0]$, and $c_j[0]$, $[0]$-Reeb chords, and $b_j[\pm 1]$, $a[\pm 1]$, and $c_j[\pm 1]$, $[\pm 1]$-Reeb chords. As above we perturb Φ_f^δ slightly in the region $|x_0| < 1$ to make it generic with respect to holomorphic disks. Note that the x_0-coordinate of $[\pm 1]$-Reeb chord equals ± 1 and that the x_0-coordinate of a $[0]$-Reeb chord is very close to 0 for small $\delta > 0$. Consider a sequence of functions f_k as above with $f_k \to 1$ as $k \to \infty$ (i.e. each f_k has a non-degenerate maximum at 0 and non-degenerate local minima at ± 1). Fix k and pick $\delta > 0$ sufficiently small so that $\Phi_{f_k}^\delta$ satisfies Lemma 4.17. Let $\Phi_k^\delta = \Phi_{f_k}^\delta$.

We next note that as $\delta \to 0$, $\Phi_k^\delta \to \Phi_k^0$ where

$$\Phi_k^0(t, q) = \Big(t, f_k'(t)z(q), x(q), f_k(t)y(q), f_k(t)z(q) \Big),$$

with $(x(q), y(q), z(q)) = \phi_0(q)$.

Lemma 4.18. *There exists k_0 such that for all $k > k_0$ there exists a $\delta_k > 0$ such that for all $\delta < \delta_k$ and any Reeb chord c the following holds. The moduli spaces $\mathcal{M}(c[0], c[1])$ and $\mathcal{M}(c[0], c[-1])$ of holomorphic disks with boundary on Φ_k^δ consists of exactly one point which is a transversely cut out rigid disk. Moreover the sign of the rigid disk in $\mathcal{M}(c[0], c[1])$ and that of the disk in $\mathcal{M}(c[0], c[-1])$ are opposite.*

Proof. First consider the case $\delta = 0$. It is easy to find rigid disks in the (x_0, y_0)-plane with positive puncture at $c[0]$ and negative puncture at $c[\pm 1]$. Moreover, by [6] Lemma 6.25 these disks are transversely cut out. To see that these are the only disks, let U and V be neighborhoods of the endpoints of the Reeb chord c in L_0 and consider the projections of $\Phi_k^0([-1, 1] \times U)$ and $\Phi_k^0([-1, 1] \times V)$ to $\mathbb{C}^n$. For sufficiently large k, these projections intersect only at 0 and it follows that there exists a positive $h > 0$ such that the area of the projection of any disk with boundary on Φ_k^0, positive puncture at $c[0]$, and negative at $c[\pm 1]$ is either equal to zero or larger than h. Since $\mathcal{Z}(c[0]) \to \mathcal{Z}(c[\pm 1])$ as $k \to \infty$ it follows that for k large enough the disks in the (x_0, y_0)-plane are the only ones. We also check the statement about signs in the case $\delta = 0$. To this end, note that the trivialized boundary conditions of the two disks in the (x_0, y_0)-plane are identical and that multiplication by -1 is a holomorphic automorphism relating them. Since multiplication by -1 reverses the orientation of the kernel of the linearized problem, it follows that their signs are opposite. Finally, we note that the fact that the moduli space $\mathcal{M}(c[0], c[\pm 1])$ corresponding to Φ_k^0 is transversely cut out implies that the statement of the lemma holds also for Φ_k^δ for all sufficiently small δ (where the smallness depends on k). $\square$

We next note that as $k \to \infty$, Φ_k^0 approaches the Legendrian submanifold

$$\Phi(t, q) = (t, 0, x(q), y(q), z(q)).$$

The projection of this Legendrian submanifold to $\mathbb{C}$ is simply the x_0-axis and its projection to $\mathbb{C}^n$ agrees with that of L_0.

Lemma 4.19. *There exists k_0 such that for all $k > k_0$ there exists a $\delta_k > 0$ such that for all $\delta < \delta_k$ and any Reeb chord $c \neq a$ the following holds. If the moduli space $\mathcal{M}_A(c[0]; \mathbf{e})$, where $\mathbf{e}$ is a word constant in the $[0]$-generators and $\mathbf{e} \neq c[\pm 1]$, has formal dimension 0 then it is empty.*

Proof. Again we start with the case $\delta = 0$. Consider a disk u as above with boundary on Φ_k^0. As $k \to \infty$, $\Phi_k^0 \to \Phi$ and the projection of u converges to a broken disk $\{v^j\}_{j=1}^m$ with boundary on L_0. The components v^j of such a broken disk either have formal dimension at least 0, or equals the handle slide disk. Also, any Reeb chord b appearing as a puncture of some v^j has $\mathcal{Z}(b) \leq \mathcal{Z}(c)$ and exactly one component of the broken disk must have its positive puncture at $c \neq a$. This component has formal dimension at least 0 (since it is not

the handle slide disk). For a disk v^j let $|v_+^j|$ is the grading of its positive puncture and $|v_-^j|$ the sum of gradings of its negative punctures and the negative of the grading of the homology data. Then the formal dimension of (the moduli space of) v^j is $|v_+^j| - |v_-^j| - 1$. The above implies that

$$N = \sum_{j=1}^{m} (|v_+^j| - |v_-^j|) \geq 1.$$

Since the positive puncture of u is its only [0]-puncture it follows that the formal dimension of u equals N. The statement of the lemma follows for $\delta = 0$. Since emptyness of a moduli space is an open condition the lemma follows in general. $\square$

Let Ω be the map from $\mathcal{A}(\Phi_k^\delta)$ to $\mathcal{A}(L_0)$ which maps $c[\pm 1]$ to c and $c[0]$ to 0 for any Reeb chord c of L.

Lemma 4.20. *There exists k_0 such that for all $k > k_0$ there exists a $\delta_k > 0$ such that for all $\delta < \delta_k$ the following holds. If u is a holomorphic disk with boundary on Φ_k^δ in $\mathcal{M}_C(a[0], \mathbf{e})$, where $\mathbf{e}$ is a word constant in the [0]-generators and $\mathbf{e} \neq a[\pm 1]$, and if this moduli space has formal dimension 0 then $C = A$ and $\Omega\mathbf{e} = \mathbf{b}$.*

Proof. Consider first the case $\delta = 0$. Taking the limit as $k \to \infty$ and arguing as in the proof of Lemma 4.19 we see that the projection of u converges to a broken disk $\{v^j\}_{j=1}^m$, that all Reeb chords b appearing as a puncture of some v^j satisfies $\mathcal{Z}(a) \geq \mathcal{Z}(b)$, and that there is a unique component with its positive puncture at a. If this component is not the handle slide disk then the argument in the proof of Lemma 4.19 shows that the formal dimension of u is at least 1. If, on the other hand, this component is the handle slide disk then the formal dimension of u equals 0 only if the broken disk has no other components. This shows the lemma for $\delta = 0$. Again since the condition that a moduli space is empty is open the lemma follows in general. $\square$

Fix k sufficiently large and $\delta > 0$ sufficiently small so that Lemmas 4.18, 4.19, and 4.20 holds for Φ_k^δ. We also assume that Φ_k^δ is generic with respect to holomorphic disks. Let $\Phi = \Phi_k^\delta$ Let $\hat{\mathcal{A}} = \mathcal{A}(\Phi)$. We denote the differential of $\hat{\mathcal{A}}$ by Δ, see Lemma 4.16. There are natural inclusions $\mathcal{A}_\pm = \mathcal{A}(L_{\pm\delta}) \subset \hat{\mathcal{A}}$. Lemma 4.14 implies that this is an inclusion of DGA's in other words,

$$\Delta c[\pm 1] = \Gamma_\pm(\partial_\pm c),$$

where $\Gamma_\pm : \mathcal{A}_\pm \to \hat{\mathcal{A}}$ is the map defined on generators by $\Gamma_\pm(c) = c[\pm 1]$, and where $\partial_\pm$ is the differential on $\mathcal{A}_\pm$. For generators $b_j[0]$ we have by Lemmas 4.18 and 4.19

(4.11) $$\Delta b_j[0] = b_j[1] - b_j[-1] + \beta_1^j + \mathcal{O}(2),$$

where β_1^j is linear in the $c[0]$-generators and $\mathcal{O}(2)$ denotes a linear combination of monomials which are at least quadratic in the [0]-generators. For the generator $a[0]$ we have by Lemmas 4.18 and 4.20

(4.12) $$\Delta a[0] = a[1] - a[-1] + \epsilon + \alpha_1 + \mathcal{O}(2),$$

where $\Omega(\epsilon) = mA\mathbf{b}$, where $m \in \mathbb{Z}$ and where α_1 is linear in the [0]-generators. For generators $c_j[0]$ we have by Lemmas 4.18 and 4.19

(4.13) $$\Delta c_j[0] = c_j[1] - c_j[-1] + \gamma_1^j + \delta_1^j(a[0]) + \mathcal{O}(2),$$

where $\gamma_1^j + \delta_1^j(a[0])$ is linear in the [0]-generators, where $\delta_1^j(a[0])$ lies in the ideal generated by $a[0]$, and where γ_1^j is constant in the $a[0]$ generator. Below we will consider ∂_+ and ∂_- as

different differentials on the algebra $\mathcal{A}$. Let ϵ be as in (4.12) and write $\theta = \Omega(\epsilon)$. Consider the tame isomorphism ψ of $\mathcal{A}$ defined on generators as

$$\psi(c) = \begin{cases} c & \text{if } c \neq a, \\ a + \theta & \text{if } c = a. \end{cases}$$

If $v \in A$ and c is a generator of $\mathcal{A}$ then let

$$\left(\frac{v}{c}\right) \bullet : \mathcal{A} \to \mathcal{A}$$

be the map defined on monomials by replacing each occurrence of c by v. Then with $\partial_+^\psi = \psi^{-1} \circ \partial_+ \circ \psi$ denoting the induced differential, a straightforward calculation gives

$$(4.14) \qquad \partial_+^\psi(c) = \begin{cases} \left(\frac{a-\theta}{a}\right) \bullet (\partial_+ c) & \text{if } c \neq a, \\ \partial_+ a + \partial_+ \theta & \text{if } c = a. \end{cases}$$

Lemma 4.21. *The algebra* $(\mathcal{A}, \partial_-)$ *is isomorphic to the algebra* $(\mathcal{A}, \partial_+^\psi)$.

Proof. We prove that the two differentials agree on generators. By Lemma 4.16, $\Delta^2 = 0$. Thus, summing the terms constant in the [0]-generators after acting by Δ in (4.11) we find

$$(4.15) \qquad 0 = \partial_+ b_j[1] - \partial_- b_j[-1] + (\Delta\beta_1)_0,$$

where $(\Delta\gamma_1)_0$ denotes the part of $\Delta\gamma_1$ which is constant in the [0]-generators. Since the constant part of $\Delta b_k[0]$ equals $b_k[1] - b_k[-1]$ it follows that

$$\Omega(\Delta\beta_1)_0 = 0.$$

Therefore, applying Ω in (4.15), we conclude

$$(4.16) \qquad \partial_- b_j = \partial_+ b_j = \left(\frac{a-\theta}{a}\right) \bullet (\partial_+ b_j),$$

since no monomial in $\partial_+ b_j$ contains a. Applying Δ to (4.12) we find similarly

$$(4.17) \qquad \partial_- a[-1] = \partial_+ a[1] + \Delta\epsilon + (\Delta\alpha_1)_0.$$

The first equality in (4.16) implies that

$$\Omega(\Delta\epsilon) = \partial_+ \theta.$$

Since every [0]-generator in α_1 is for the form $b_j[0]$ we find, as with β_1 above, that $\Omega(\Delta\alpha_1)_0 = 0$. We conclude

$$(4.18) \qquad \partial_- a = \partial_+ a + \partial_+ \theta.$$

Applying Δ to (4.13) gives

$$(4.19) \qquad \partial_- c_j[-1] = \partial_+ c_j[1] + \left(\Delta\gamma_1^j\right)_0 + \left(\Delta\delta_1^j(a[0])\right)_0.$$

Applying $\left(\frac{a[-1]-\epsilon}{a[1]}\right) \bullet$ to both sides in (4.19) and noting that no monomial in $\partial_- c_j[-1]$ contains an $a[1]$ generator we get

$$(4.20) \qquad \begin{aligned} \partial_- c_j[-1] = &\left(\frac{a[-1]-\epsilon}{a[1]}\right) \bullet (\partial_+ c_j[1]) + \left(\frac{a[-1]-\epsilon}{a[1]}\right) \bullet \left(\Delta\gamma_1^j\right)_0 \\ &+ \left(\frac{a[-1]-\epsilon}{a[1]}\right) \bullet \left(\Delta\delta_1^j(a[0])\right)_0. \end{aligned}$$

Each term in $\left(\Delta\delta_1^j(a[0])\right)_0$ arises by replacing $a[0]$ in every monomial $\xi a[0]\eta$ of $\delta_1^j(a[0])$ with $(a[1] - a[-1] + \epsilon)$ yielding $\xi(a[1] - a[-1] + \epsilon)\eta$. When $\left(\frac{a[-1]-\epsilon}{a[1]}\right)\bullet$ is a applied to $\xi(a[1] - a[-1] + \epsilon)\eta$ the result is

$$\xi(a[-1] - \epsilon - a[-1] + \epsilon)\eta = 0.$$

Thus, the last term in (4.20) vanishes. Since the $[0]$-generator of any monomial in γ_1^j equals either $c_k[0]$ for some k, or $b_r[0]$ for some r and since the constant part of $\Delta c_k[0]$ equals $c_k[1] - c_k[-1]$, and the constant part of $\Delta b_k[0]$ equals $b_k[1] - b_k[-1]$, we conclude that

$$\Omega\left(\frac{a[-1]-\epsilon}{a[1]}\right)\bullet\left(\Delta\gamma_1^j\right)_0 = 0.$$

Thus, applying Ω to (4.20) we arrive at

(4.21)
$$\partial_- c_j = \left(\frac{a-\theta}{a}\right)\bullet\partial_+ c_j.$$

The lemma follows from (4.16), (4.18), (4.21). $\square$

Corollary 4.22. *If $L_t \subset \mathbb{C}^n \times \mathbb{R}$, $-1 \leq t \leq 1$, is a Legendrian isotopy with a generic handle slide at $t = 0$ as above then the stable tame isomorphism classes of $(\mathcal{A}(L_{-1}, \partial_{-1}))$ and $(\mathcal{A}(L_1), \partial_1)$ are the same.* $\square$

4.3.4. *Invariance under self tangencies.* We now turn our attention to self tangency instances. As shown in [6] there is no loss of generality in assuming that our isotopy near the self tangency instant has standard from. Consider a 1-parameter family of Legendrian submanifolds $L_t \subset \mathbb{C}^n \times \mathbb{R}$, $t \in [-1, 1]$. We assume that L_t is constant in t outside $B_{2r}(0) \times \mathbb{R}$ for some ball $B_{2r}(0) \subset \mathbb{C}^n$ or radius $2r$ around the origin. The intersection of L_t with $B_r(0) \times \mathbb{R}$ consists of two sheets L_t^+ and L_t^-. Let $(x, y, z) = (x + iy, z)$, $x, y \in \mathbb{R}^n$, and $z \in \mathbb{R}$ be coordinates on $\mathbb{C}^n \times \mathbb{R}$. Assume that the sheet L_t^- is constant in t and that it satisfies

$$L_t^- = B_{2r}(0) \times \mathbb{R} \cap \{y = 0, z = 0\}.$$

The second sheet is moving with t. Let $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, and let $\langle , \rangle$ denote the standard inner product on $\mathbb{R}^k$. Then $L_t^+ \cap B_r(0) \times \mathbb{R}$ is given by the map $[-1, 1] \times \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{C}^n \times \mathbb{R}$, $(t, q_1, q_2) \mapsto (x_1 + iy_1, x_2 + iy_2, z)$ where

$$x_1(t, q_1, q_2) = q_1,$$
$$y_1(t, q_1, q_2) = 3q_1^2 + t,$$
$$z(t, q_1, q_2) = q_1(q_1^2 + t) + c + \langle q_2, q_2 \rangle,$$
$$x_2(t, q_1, q_2) = q_2,$$
$$y_2(t, q_1, q_2) = 2q_2,$$

where $c > 0$ is a constant. In the region $L_t^+ \cap (B_{2r}(0) \setminus B_r(0)) \times \mathbb{R}$ the isotopy interpolates by the one above and the constant isotopy. We denote this 1-parameter family of Legendrian embeddings $\phi_t : L \to \mathbb{C}^n \times \mathbb{R}$, $t \in [-1, 1]$. Let $\delta > 0$. Note that $\phi_{-\delta}$ has two Reeb chords more than ϕ_δ and that the extra Reeb chords of $\phi_{-\delta}$ converges to the self tangency Reeb chord o of ϕ_0. We choose notation so that the Reeb chords of ϕ_δ are

$$\{b_1, \ldots, b_s, a_1, \ldots, a_l\},$$

those of ϕ_0 are

$$\{b_1, \ldots, b_s, o, a_1, \ldots, a_l\},$$

and those of $\phi_{-\delta}$ are

$$\{b_1, \ldots, b_s, b, a, a_1, \ldots, a_l\},$$

and so that

$$\mathcal{Z}(b_1) \leq \cdots \leq \mathcal{Z}(b_r) \leq \mathcal{Z}(b) \leq \mathcal{Z}(o) \leq \mathcal{Z}(a) \leq \mathcal{Z}(a_1) \leq \cdots \leq \mathcal{Z}(a_s).$$

Fix a positive Morse function f with local minima at ± 1 and no critical points in $(-\infty, -1) \cup (1, \infty)$ and with one local maximum at $\beta \in (0, 1)$, for some small β. Assume that $\phi_0(L)$ is a generic self tangency moment which means that all moduli spaces of negative formal dimension are empty and that all rigid disks with boundary on $\phi_0(L)$ are transversely cut out.

Lemma 4.23. *Fix f as above. Then there exists δ_0 such that for all $\delta < \delta_0$ the Reeb chords of Φ_f^δ are*

$$\{b_j[-1], b_j[1], b_j[0]\}_{j=1}^r \cup \{b[-1], a[-1]\} \cup \{a_j[-1], a_j[1], a_j[0]\}_{j=1}^s,$$

where the x_0-coordinates of the $[\pm 1]$-Reeb chords are ± 1, and where the x_0-coordinates of the $[0]$-Reeb chords are close to 0.

Proof. Consider first the case $\delta = 0$. The Reeb chords of Φ_f^0 are easily seen to be

$$\{b_j[-1], b_j[1], b_j[0]\}_{j=1}^r \cup \{o[-1], o[1], o[0]\} \cup \{a_j[-1], a_j[+1], a_j[0]\}_{j=1}^s,$$

and all except $o[\pm 1]$ and $o[0]$ have transverse tangent planes at their ends. We conclude that for $\delta > 0$ sufficiently small Φ_f^δ has the Reeb chords

$$\{b_j[-1], b_j[1], b_j[0]\}_{j=1}^r \cup \{a_j[-1], a_j[+1], a_j[0]\}_{j=1}^s,$$

and possibly some Reeb chords in a neighborhood of $o[\pm 1]$ and of $o[0]$. To find these we take a closer look at Φ_f^δ. If o^- denotes the lower endpoint of o then in a neighborhood of $\{o^-\} \times \mathbb{R}$, Φ_f^δ is simply the embedding (with notation as above)

$$\Phi(t, q) = (t, 0, q, 0, 0).$$

In neighborhoods of $(\pm 1, \{o^+\})$, Φ_f^δ is given by

$$
\begin{aligned}
x_0(t, q_1, q_2) &= t, \\
y_0(t, q_1, q_2) &= f'(t)\Big(q_1(q_1^2 \pm \delta) + c + \langle q_2, q_2 \rangle\Big) \\
x_1(t, q_1, q_2) &= q_1, \\
y_1(t, q_1, q_2) &= f(t)(3q_1^2 \pm \delta), \\
x_2(t, q_1, q_2) &= q_2, \\
y_2(t, q_1, q_2) &= f(t)2q_2 \\
z(t, q_1, q_2) &= f(t)\Big(q_1(q_1^2 \pm \delta) + c + \langle q_2, q_2 \rangle\Big),
\end{aligned}
$$

and we find that there are two Reeb chords $a[-1]$ and $b[-1]$ in a neighborhood of $o[-1]$ and no Reeb chords in a neighborhood of $o[1]$. In a neighborhood of $(0, \{o^+\})$ it is given by

$$x_0(t, q_1, q_2) = t,$$
$$y_0(t, q_1, q_2) = f'(t)\Big(q_1(q_1^2 + \delta t) + c + \langle q_2, q_2 \rangle\Big) + f(t)\delta q_1$$
$$x_1(t, q_1, q_2) = q_1,$$
$$y_1(t, q_1, q_2) = f(t)(3q_1^2 + \delta t),$$
$$x_2(t, q_1, q_2) = q_2,$$
$$y_2(t, q_1, q_2) = f(t)2q_2$$
$$z(t, q_1, q_2) = f(t)\Big(q_1(q_1^2 + \delta t) + c + \langle q_2, q_2 \rangle\Big),$$

To have a Reeb chord we note first that the equation $3q^2 + \delta t = 0$ must hold. Since $\delta > 0$, the t-coordinate of any Reeb chord thus satisfies $t < 0$. Moreover, $|t| < 1$ implies that at a Reeb chord $q_1 = \mathcal{O}(\delta^{\frac{1}{2}})$ and from the expression for y_2, $q_2 = 0$. The final condition is $y_0 = 0$, which implies

$$f'(t)(c + \mathcal{O}(\delta^{\frac{3}{2}})) + f(t)\mathcal{O}(\delta^{\frac{3}{2}}) = 0.$$

By the choice of f, the local maximum of f lies at $t = \beta > 0$ thus $f'(t) > 0$ for $-1 < t \leq 0$ and letting $\delta \to 0$ we see that Φ_f^δ does not have any Reeb chords near $o[0]$. This finishes the proof. $\qquad \square$

Consider a sequence of functions f_k as above with $f_k \to 1$ as $k \to \infty$ (i.e. each f_k has a non-degenerate maximum at $\beta_k > 0$ and non-degenerate local minima at ± 1). Fix k and pick $\delta > 0$ sufficiently small so that $\Phi_{f_k}^\delta$ satisfies Lemma 4.23. Let $\Phi_k^\delta = \Phi_{f_k}^\delta$.

Lemma 4.24. *There exists k_0 such that for all $k > k_0$ there exists a $\delta_k > 0$ such that for all $\delta < \delta_k$ and a Reeb chord $c \in \{b_j\}_{j=1}^r \cup \{a_k\}_{k=1}^s$ the following holds. The moduli spaces $\mathcal{M}(c[0], c[1])$ and $\mathcal{M}(c[0], c[-1])$ of holomorphic disks with boundary on Φ_k^δ consists of exactly one point which is a transversely cut out rigid disk. Moreover the sign of the rigid disk in $\mathcal{M}(c[0], c[1])$ and that of the disk in $\mathcal{M}(c[0], c[-1])$ are opposite.*

Proof. The proof is a word by word repetition of the proof of Lemma 4.18. $\qquad \square$

We next note that as $k \to \infty$, Φ_k^0 approaches the Legendrian submanifold

$$\Phi(t, q) = (t, 0, x(q), y(q), z(q)).$$

The projection of this Legendrian submanifold to $\mathbb{C}$ is simply the x_0-axis and its projection to $\mathbb{C}^n$ agrees with that of L_0.

Lemma 4.25. *There exists k_0 such that for all $k > k_0$ there exists a $\delta_k > 0$ such that for all $\delta < \delta_k$ and any Reeb chord b_j, $j = 1, \ldots, r$, the following holds. If the moduli space $\mathcal{M}_A(b_j[0]; \mathbf{e})$, where $\mathbf{e}$ is a word constant in the $[0]$-generators and $\mathbf{e} \neq b_j[\pm 1]$, has formal dimension 0 then it is empty.*

Proof. For k sufficiently large, $\mathcal{Z}(b_j[0]) < \mathcal{Z}(a[-1]) < \mathcal{Z}(b[-1])$. Therefore, for such k, a disk with its positive puncture at $b_j[0]$ must have negative punctures mapping to Reeb chords in the set $\{b_i[0], b_i[\pm 1]\}_{i=1}^s$. After this observation the lemma follows from the proof of Lemma 4.19. $\qquad \square$

Lemma 4.26. *There exists k_0 such that for all $k > k_0$ there exists a $\delta_k > 0$ such that for all $\delta < \delta_k$ and any Reeb chord a_j, $j = 1, \ldots, s$ the following holds. If the moduli space $\mathcal{M}_A(a_j[0]; \mathbf{e})$, where $\mathbf{e}$ is a word constant in the $[0]$-generators and $\mathbf{e} \neq a_j[\pm 1]$, has formal dimension 0 then $a[-1]$ appears at least once as a letter in the word $\mathbf{e}$.*

Proof. Let u be a disk in $\mathcal{M}_A(a_j[0]; \mathbf{e})$. As in the proof of Lemma 4.20 its projection to $\mathbb{C}^n$ converges to a broken disk with boundary on L_0 as $k \to \infty$. Let $\{v^j\}_{j=1}^m$ be the components of this broken disk. As above we let o denote the degenerate Reeb chord of L_0. Let $|o| = |b[-1]| = |a[-1]| - 1$. Then the formal dimension of a disk v_j satisfies the following.

- If the positive puncture of v_j does not equal o then the formal dimension of v_j equals $|v_j^+| - |v_j^-| - 1$, where $|v_j^+|$ is the grading of the Reeb chord at its positive puncture, and where $|v_j^-|$ is the sum of the gradings at its negative corners and the negative of the grading of its homology data.
- If the positive puncture of v_j equals o then the formal dimension of v_j equals $|v_j^+| - |v_j^-|$.

Since each of the disks v_j have non-negative formal dimension and since at least one of them does not have its positive puncture at o, we find

$$\sum_{j=1}^m (|v_j^+| - |v_j^-|) \geq 1.$$

Thus, assuming that none of the negative punctures of u map to $a[-1]$ we find that the formal dimension of u is at least 1 which contradicts it being rigid. It follows that at least one of the negative punctures of u map to $a[-1]$. $\square$

Fix k sufficiently large and $\delta > 0$ sufficiently small so that Lemmas 4.25 and 4.26 hold for Φ_k^δ. Let $\Phi = \Phi_k^\delta$. Consider the algebra $\hat{\mathcal{A}} = \mathcal{A}(\Phi)$ and let its differential be Δ. Note that $\mathcal{A}_+ = \mathcal{A}(L_\delta)$ and $\mathcal{A}_- = \mathcal{A}(L_{-\delta})$ can be considered as subalgebras of $\hat{\mathcal{A}}$ via the map which takes a_j and b_j to $a_j[\pm 1]$ and $b_j[\pm 1]$ and a and b to $a[-1]$ and $b[-1]$, respectively. It is a consequence of Lemma 4.14 that $\mathcal{A}_\pm$ are differential graded subalgebras of $\hat{\mathcal{A}}$. In other words, denoting their respective differentials ∂_+ and ∂_- we have

$$\Delta(b_j[\pm 1]) = \Gamma_\pm(\partial_\pm b_j), \quad \text{for } j = 1, \dots, r,$$
$$\Delta(a[-1]) = \Gamma_-(\partial_- a),$$
$$\Delta(b[-1]) = \Gamma_-(\partial_- b),$$
$$\Delta(a_j[\pm 1]) = \Gamma_\pm(\partial_\pm a_j), \quad \text{for } j = 1, \dots, s,$$

where $\Gamma_\pm a_j = a_j[\pm 1]$, $\Gamma_\pm b_j = b_j[\pm 1]$, $\Gamma_- a = a[-1]$, and $\Gamma_- b = b[-1]$. It follows from Lemma 4.25 that

$$(4.22) \qquad \qquad \Delta b_j[0] = b_j[1] - b_j[-1] + \beta_1^j + \mathcal{O}(2),$$

where β_1^j denotes term which is linear in the $[0]$-generators and $\mathcal{O}(2)$ the term which is quadratic and higher. It follows from Lemma 4.26 that

$$(4.23) \qquad \qquad \Delta a_j[0] = a_j[1] - a_j[-1] + \gamma(a[-1]) + \alpha_1^j + \mathcal{O}(2),$$

where $\gamma(a[-1])$ lies in the ideal generated by $a[-1]$ and is constant in the $[0]$-generators and where α_1^j is the linear in the $[0]$-generators. Consider the stabilized algebra $S(\mathcal{A}_+)$ with extra generators e_0 and e_1, $|e_0| = |a|$ and $|e_1| = |b| = |a| - 1$ and define the algebra homomorphism $\Phi_0 \colon \mathcal{A}_- \to S(\mathcal{A}_+)$

$$\Phi_0(c) = \begin{cases} e_0 & \text{if } c = a, \\ e_1 - v & \text{if } c = b, \\ c & \text{otherwise}, \end{cases}$$

where v is the unique element in $\mathcal{A}_+$ such that $\partial_- a = b + v$ (in $\mathcal{A}_-$). For the existence of such an element see [6], Lemma 1.16. Let $\tau \colon S(\mathcal{A}_+) \to \mathcal{A}_+$ be the natural projection and let $\mathcal{A}_j \subset \mathcal{A}_-$ be the subalgebra generated by $\{b_1, \dots, b_s, b, a, a_1, \dots, a_j\}$. Then

Lemma 4.27.

(4.24) $$\Phi_0 \circ \partial_- w = \partial_+^s \circ \Phi_0 w,$$

for all $w \in \mathcal{A}_0$ and

(4.25) $$\tau \circ \Phi_0 \circ \partial_- = \tau \circ \partial_+^s \circ \Phi_0.$$

Proof. Let Ω be the map which takes $b_j[-1]$ and $b_j[1]$ to b_j and which takes $a_j[1]$ and $a_j[-1]$ to a_j. Then, for generators b_j, (4.24) follows by applying $\Omega \circ \Delta$ to (4.22). We also have

$$\Phi_0 \partial_- b = \Phi_0(-\partial_- v) = \partial_+^s \Phi_0 b,$$

and

$$\Phi_0 \partial_- a = \Phi_0(b + v) = e_1 = \partial_+ \Phi_0 a.$$

Thus, (4.24) holds and it is sufficient to prove (4.25) for a_j-generators to conclude it holds in general. Applying Δ to (4.23) and considering the constant term we find

$$\partial_+ a_j[1] = \partial_- a_j[-1] - \Delta(\gamma(a[-1])) - (\Delta \alpha_1^j)_0.$$

Letting $\tilde{\Phi}_0$ be the map which takes $a[-1]$ to e_0 and $b[-1]$ to $e_1 - \Gamma_-(v)$ we find that

$$\partial_+ a_j[1] = \tilde{\Phi}_0 \partial_- a_j[-1] - \tilde{\Phi}_0(\Delta(\gamma(a[-1]))) - \tilde{\Phi}_0(\Delta \alpha_1^j)_0.$$

Since the constant part of $\Delta b_j[0]$ equals $(b_j[1] - b_j[-1])$ we note that Ω annihilates each polynomial in $(\Delta \alpha_1^j)_0$ which originates from a monomial in α_1^j in the ideal generated by $b_j[0]$. Moreover, the constant part of $\Delta a_j[0]$ equals $a_j[1] - a_j[-1] + \gamma(a[-1])$. Since τ annihilates $\Phi_0(\gamma(a[-1]))$ and since Ω annihilates $a_j[1] - a_j[-1]$ and since τ and Ω commutes we find that $\tau \circ \Omega$ annihilates the last term. Finally, each term in $(\Delta(\gamma(a[-1]))$ contains either $a[-1]$ or (when $a[-1]$ is differentiated) $b[-1] + \Gamma_- v$. Hence, τ annihilates $\Phi_0(\Delta \gamma(a[-1]))$ and we conclude

$$\tau \partial_+ a_j = \tau \Phi_0 \partial_- a_j,$$

as claimed. □

Lemma 4.28. $\mathcal{A}(L_-)$ *is stable tame isomorphic to* $\mathcal{A}(L_+)$.

Given Lemma 4.27 the proof of this lemma is standard, see [3, 6, 9].

4.4. **Change of spin structure.** In our construction of Legendrian contact homology in Subsection 4.1 we assume that our Legendrian manifolds $L \subset \mathbb{C}^n \times \mathbb{R}$ are spin and we fix a spin structure on L. In general the DGA $(\mathcal{A}(L), \partial)$ of L depends on the fixed spin structure. We explain here the exact form of this dependence.

4.4.1. *Spin structures.* Let M be an oriented n-manifold which is spin. In Subsection 3.4 we viewed spin structures as trivializations of $\tilde{T}M$ over the 1-skeleton that extends over the 2-skeleton. Given a spin structure $\mathfrak{s}_0$ on M we can identify the set of spin structures $\mathrm{Spin}(M)$ with $H^1(M; \mathbb{Z}_2)$. Specifically, let $\mathfrak{s}$ be another spin structure. Denote the trivialization associated to $\mathfrak{s}$ by $\tau_{\mathfrak{s}}$. Isotop $\tau_{\mathfrak{s}_0}$ and $\tau_{\mathfrak{s}}$ to be the same on the 0-skeleton $\Delta^{(0)}$ of M. Now along each edge in the 1-skeleton $\Delta^{(1)}$ of M we can compare $\tau_{\mathfrak{s}_0}$ to $\tau_{\mathfrak{s}}$. This gives us a loop in $\pi_1(SO(n+2)) = \mathbb{Z}_2$. Thus we have a map $d(\mathfrak{s}_0, \mathfrak{s}) : \Delta^{(1)} \to \mathbb{Z}_2$. One may easily check that this gives a well defined cohomology class $[d(\mathfrak{s}_0, \mathfrak{s})] \in H^1(M; \mathbb{Z}_2)$. It is a standard fact that

$$d(\mathfrak{s}_0, \cdot) : \mathrm{Spin}(M) \to H^1(M; \mathbb{Z}_2)$$

is a one to one correspondence. We denote this map $d_{\mathfrak{s}_0}$.

4.4.2. *The change of the differential.* Let $L \subset \mathbb{C}^n \times \mathbb{R}$ be a Legendrian submanifold with a fixed spin structure $\mathfrak{s}_0$. Let $\mathfrak{s}$ be another spin structure on L. Let $\partial_{\mathfrak{s}_0}$ and $\partial_{\mathfrak{s}}$ denote the differentials on $\mathcal{A}(L)$ induced from the spin structures $\mathfrak{s}_0$ and $\mathfrak{s}$ respectively. Let $A \in H_1(L)$.

Theorem 4.29. *Let a be any Reeb chord of L. Assume that*

$$\partial_{\mathfrak{s}_0} a = \sum_j m_j A_j \mathbf{b}_j,$$

where $m_j \in \mathbb{Z}$, $A_j \in H_1(L)$, and where $\mathbf{b}_j$ is a Reeb chord word. Then

$$\partial_{\mathfrak{s}} a = \sum_j \sigma(\langle d_{\mathfrak{s}_0}(\mathfrak{s}), A_j \rangle) m_j A_j \mathbf{b}_j,$$

where $\sigma \colon \mathbb{Z}_2 \to \{1, -1\}$ is the non-trivial homomorphism.

Proof. Take a triangulation of L containing all the double points in the complex projection in the 0-skeleton and containing the capping paths for the double points in the 1-skeleton. (We can assume the capping paths are all disjoint arcs in $\Delta^{(1)}$.) Note that we may find trivializations $\tau_{\mathfrak{s}_0}$ and $\tau_{\mathfrak{s}}$ corresponding to $\mathfrak{s}_0$ and $\mathfrak{s}$, respectively, of $\tilde{T}L$ over $\Delta^{(2)}$ such that the trivializations agree over all capping paths. Let u be any rigid holomorphic disk. Consider the trivializations of the Lagrangian boundary conditions on the closed disk which arises when u is capped off and which is induced from $\tau_{\mathfrak{s}_0}$ and $\tau_{\mathfrak{s}}$ respectively. The difference of these trivializations arises in differences of the framings along the boundary paths of u. Since $\tau_{\mathfrak{s}_0}$ and $\tau_{\mathfrak{s}}$ agrees on the capping paths it follows that this framing difference is exactly $\sigma(\langle d_{\mathfrak{s}_0}(\mathfrak{s}), A \rangle)$, where A is the homology class encoding the boundary of u. By [12] the canonical orientation on the determinant line of homotopically different trivializations of a Lagrangian boundary condition on the closed disk are opposite. The lemma follows. $\square$

4.4.3. *Change in the DGA.* While it seems crucial that a Legendrian submanifold be spin to define an oriented version of contact homology, there is no real dependence on the spin structure, when we define contact homology over $\mathbb{Z}[H_1(L)]$.

Theorem 4.30. *Let L be a Legendrian submanifold of $\mathbb{R}^{2n+1}$. The DGA's of L over $\mathbb{Z}[H_1(L)]$ associated to any two spin structures are tame isomorphic.*

Proof. Given two spin structures $\mathfrak{s}$ and $\mathfrak{s}'$. Let $\mathcal{A}$ and $\mathcal{A}'$ be the DGA's associated to L using the two spin structures. Using the notation from Theorem 4.29 define $\phi : \mathcal{A} \to \mathcal{A}'$ to be the identity of the generators of the DGA but for $A \in H_1(L)$ let $\phi(A) = \sigma(\langle d_{\mathfrak{s}}(\mathfrak{s}'), A \rangle) A$ (thus the isomorphism ϕ arises from an isomorphism of the base ring $\mathbb{Z}[H_1(L)]$). One may easily check that ϕ is a chain map and hence a tame isomorphism from $\mathcal{A}$ to $\mathcal{A}'$. $\square$

It is quite interesting to note that if one merely uses orientations to define the DGA for L over $\mathbb{Z}$ then there is a dependence on spin structures.

Theorem 4.31. *The DGA's of a Legendrian submanifold L defined over $\mathbb{Z}$ using two different spin structures are not necessarily stable tame isomorphic.*

Proof. If the DGA's are stable tame isomorphic then the contact homology associated to L with the two spin structures would be isomorphic. Let L be the Legendrian unknot in $\mathbb{R}^3$ whose Lagrangian projection has one double point. Thus $\mathcal{A} = \mathbb{Z}\langle a \rangle$, where a is the double point in the projection. If we use the spin structure on L defined in Subsection 4.5 then $\partial a = 2$ so the contact homology is $\mathbb{Z}_2$. By Theorem 4.29 we see that using the other spin structure on L will give a differential $\partial a = 0$. So the contact homology with this spin structure is a copy of $\mathbb{Z}$ in gradings 0 and 1. Thus the DGA's associated to L using the two spin structures are not stable tame isomorphic. To get examples in higher dimensions one can use the spinning construction, see [5]. $\square$

4.5. **The three dimensional case.** In this section we show that with the proper choice of spin structure on S^1 the DGA we associate to a Legendrian knot in $\mathbb{R}^3$ is the same as the combinatorially defined one given in [9]. We also deduce an alternative combinatorial description and demonstrate that these two in a certain sense constitute a complete list of possible combinatorial definitions. Recall that our construction of orientations on the moduli spaces relevant to contact homology depend on choices. Specifically (see Section 3.3) we chose an orientation on $\mathbb{R}^n$, on the capping operators, on spaces of conformal structures $\mathcal{C}_m$ and automorphisms $\mathcal{A}_m$, and on $\mathbb{C}$. This last choice was largely hidden in previous sections: we were simply using the natural complex orientation $\mathbb{C}$. However, there is no real need to choose this orientation and, as we shall see, the choice matters. We will call the choices listed above *the choice of basic orientations*.

4.5.1. *Combinatorial descriptions.* Let $L \subset \mathbb{C} \times \mathbb{R}$ be an oriented Legendrian knot and consider a double point of its Lagrangian projection. Near this double point the Lagrangian projection subdivides the plane into four quadrants. We describe two shading rules.

(A) A quadrant of a double point with even grading (see section 4.2) is *A-shaded* if it is adjacent to the incoming edge of the overcrossing, the other two quadrants are *A-unshaded*. All quadrants of odd double points are *A-unshaded*. See the left hand side of Figure 1.

(B) A quadrant of a double point with even grading (see section 4.2) is *B-shaded* if it is adjacent to the incoming edge of the overcrossing and to the outgoing edge of the undercrossing, the other three quadrants are *B-unshaded*. A quadrant of an odd double points is B-shaded if it adjacent to the incoming edges of both the over- and the undercrossing, the other three are B-unshaded. See the right hand side of Figure 1.

$$\text{Even} \qquad \text{Odd} \qquad\qquad \text{Even} \qquad \text{Odd}$$

FIGURE 1. Two sign rules

The main goal of this section is to prove the following theorem.

Theorem 4.32. *Let L be an oriented Legendrian knot in $\mathbb{R}^3$ equipped with the Lie group spin structure of $L = S^1$. Then there exists a choice of basic orientations such that for any Reeb chord a of L,*

$$\partial a = \sum \left(\sum_{u \in \mathcal{M}(a; b_1 \ldots b_n)} (-1)^{s_A(u)} b_1 \ldots b_n \right),$$

where the first sum is over words $b_1 \ldots b_n$ in the double points of the Lagrangian projection of L with $|a| - |b_1| - \cdots - |b_n| = 1$ and $s_A(u)$ is the number of A-shaded corners in the image of u. Moreover, there exists another choice of basic orientations (where the orientation of $\mathbb{C}$ is opposite, for more detail see Lemma 4.40) such that

$$\partial a = \sum \left(\sum_{u \in \mathcal{M}(a; b_1 \ldots b_n)} (-1)^{s_B(u)} b_1 \ldots b_n \right),$$

where the first sum is over words $b_1 \ldots b_n$ in the double points of the Lagrangian projection of L with $|a| - |b_1| - \cdots - |b_n| = 1$ and $s_B(u)$ is the number of B-shaded corners in the image of u.

Remark 4.33. The sign rule presented in [9] is the one corresponding to the A-shading.

Remark 4.34. Note that we have not explicitly identified which orientation on $\mathbb{C}$ gives which orientation convention. We also note that other changes in the basic orientations give differentials closely related to those given here. (For reasonable definitions of basic orientations they differ by an over-all sign.)

Remark 4.35. To compute the differential with respect to the other spin structure on S^1 (the null-cobordant one) one can appeal to Theorem 4.29. However, there is also a simple way to compute the differential in this case. Start with the trivialization of the stabilized tangent bundle to S^1 corresponding to the Lie group spin structure and add a π-rotation in a small neighborhood of a point $p \in S^1$. We may assume that no capping path used in computing the algebra and differential contains p. If $u \in \mathcal{M}(a; b_1, \ldots, b_n)$ then define $I(u)$ to be the number of times $u(\partial D_{n+1})$ intersects p. As in the proof of Theorem 4.29 we see that the differential ∂' corresponding to the new spin structure is

$$\partial' a = \sum \left(\sum_{u \in \mathcal{M}(a; b_1, \ldots, b_n)} (-1)^{s_*(u) + I(u)} b_1 \ldots b_n \right),$$

where $* = A$ or $* = B$.

4.5.2. *Stabilization and trivialization.* Recall from Subsection 3.4 that when given a chord generic Legendrian knot $L \subset \mathbb{C} \times \mathbb{R}$ we first consider the Lagrangian projection $\Pi_{\mathbb{C}}$ of L to $\mathbb{C}$ then the inclusion of $\mathbb{C}$ into $\mathbb{C}^3$ as the first coordinate. In the 1-dimensional case the complex angle has only one component and therefore a 1-dimensional stabilization is sufficient. We thus consider a $\mathbb{C}^2$ bundle over L and a field of Lagrangian subspaces of the bundle by assigning to each point p in L the Lagrangian subspace

(4.26) $(t(p), e^{i\theta(p)})$

where $t(p) = \Pi_{\mathbb{C}}(\tau_p)$ for the unit tangent vector τ_p to L at p and where $\theta \colon L \to [-\theta_0, \theta_0]$, for some small θ_0, and $\theta(p) = \theta_0$ for p in a neighborhood of each upper end of a Reeb chord and $\theta(p) = -\theta_0$ in a neighborhood of each lower end point of a Reeb chord. The auxiliary linearized problem for a holomorphic disk $u \colon D_{m+1} \to \mathbb{C}$ is then the $\bar{\partial}$-problem with boundary condition given by the above plane field along $u(\partial D_{m+1})$. We note that this auxiliary linearized problem is split: $\bar{\partial} = \bar{\partial}_1 \oplus \bar{\partial}_2$ where $\bar{\partial}_j$ acts on the jth coordinate of a section. Moreover,

$$\text{Index}(\bar{\partial}_2) = 0,$$

and $\bar{\partial}_2$ is an isomorphism. To fit the above into the orientation scheme presented in previous sections, we need a trivialization of the Lagrangian plane field which meet the conditions presented in Subsection 3.4.2 at the double points of L. (Note that (4.26) gives a trivialization which does not necessarily meet the conditions at Reeb chords.) To achieve this we change the trivialization in (4.26) in a neighborhood of the upper end of each *odd* Reeb chord as follows. Following the orientation of the knot we add a π-rotation to the trivialization right before we come to the upper Reeb chord end and a $(-\pi)$-rotation right after (here we think of the Lagrangian plane field as oriented by the trivialization presented above). With this modification the trivialization does meet the necessary conditions and can be capped off with R_{ne}, R_{no}, R_{pe}, and R_{po}.

4.5.3. *Two bundles.* Consider the space X_{m+1} of holomorphic immersions $D_{m+1} \to \mathbb{C}$ with $(m+1)$ convex corners on the boundary. One called positive and the rest called negative.

Lemma 4.36. *The weak homotopy type of X_{m+1} equals that of $SO(2)$. (That is X_{m+1} is a $K(\mathbb{Z};1)$-space.)*

Proof. Each disk immersion with a marked point on its boundary can be contracted through immersed disks to a small standard immersed disk near its marked point. In the present set up we use the positive corner as the marked point and note that there is no problem keeping control of the rest of the corners during such a deformation (parameterized by a compact space). $\square$

We associate to each element $u \in X_{m+1}$ a Lagrangian boundary condition Λ_u for the $\bar{\partial}$-operator on $\mathbb{C}^2$-valued functions on D_{m+1} by defining, with τ_ζ denoting the positive tangent vector of ∂D_{m+1} at ζ,

$$\Lambda(\zeta) = \mathrm{Span}(du(\tau_\zeta), e^{i\theta(\zeta)}) \subset \mathbb{C}^2,$$

where $-\theta_0 \leq \theta(\zeta) \leq \theta_0$ and where $\theta = \theta_0$ ($\theta = -\theta_0$) near the negative punctures along the incoming (outgoing) part of ∂D_{m+1}. Along the corresponding parts of ∂D_{m+1} near a positive puncture we let $\theta(\zeta)$ have the opposite signs. Note that the $\bar{\partial}$-operator just mentioned has index $-(m-2)$ and that the dimension of its kernel (cokernel if $m < 2$) equals 0 over each $u \in X_{m+1}$. Let $D \to X_{m+1}$ be the vector bundle with fiber over u equal to the cokernel (kernel) of this operator. Since the generator of $\pi_1(X_{m+1})$ can be represented by a rigid rotation of a convex disk around its positive puncture it is easy to see that D is orientable. We consider also the bundle $E \to X_{m+1}$, the fiber of which over u is the tangent space of conformal structures (automorphisms if $m < 2$) of the source D_{m+1} of u. This is a fiber bundle of fiber dimension $m - 2$. Moreover, the natural linearization map $i \circ \partial u \circ \gamma$, where $\gamma \in \mathrm{End}(TD_{m+1})$ is a variation of the conformal structure gives a fiberwise isomorphism $E \to D$ (this is a consequence of the general transversality properties of the $\bar{\partial}$-equation in dimension 1: rigid disk are automatically transversely cut out). The natural orientation of spaces of conformal structures induces an orientation on E. Assume that D is oriented. Then orientations of D and E either agrees or disagrees over every $u \in X_{m+1}$.

4.5.4. *Diagram orientations and diagram trivializations.* We consider special types of trivializations of Λ_u. These trivializations model those coming from a knot diagram as mentioned above. Let $\partial D_{m+1} = I_1 \cup \cdots \cup I_{m+1}$, be a subdivision into connected components where I_1 (I_{m+1}) has its negative (positive) end at the positive puncture. For $u \in X_{m+1}$ we define a *diagram orientation* of $u^*T\mathbb{C}$ over ∂D_{m+1} to be a trivialization of this pull back bundle which has the form

$$t(\zeta) = \pm du(\tau_\zeta), \quad \text{for } \zeta \in I_j, \text{ all } j.$$

Let p be the positive corner of u then we call p *even (odd)* if a positive rotation of the incoming trivialization vector with magnitude equal to the exterior angle of u at p gives the negative (positive) outgoing trivialization vector. Let q be a negative corner of u then we call q *even (odd)* if a positive rotation of the incoming trivialization vector with magnitude equal to the exterior angle of u at q gives the positive (negative) outgoing trivialization vector. Note that a corner is even or odd according to whether or not the corresponding Reeb chord has even or odd grading (see Lemma 4.3), thus the number of odd corners of any diagram orientation of a rigid disk in X_{m+1} is odd. We associate a *diagram trivialization* of the Lagrangian boundary condition Λ_u to a diagram orientation as follows. Fix a small neighborhood $U_j \subset I_j$ of the positive (negative) end point of each I_j which has its positive (negative) end point at a negative (positive) *odd* puncture. The restriction of the diagram

trivialization to the complement of the union of these fixed neighborhoods is simply

$$(t(\zeta), e^{i\theta(\zeta)}).$$

To complete the definition of the diagram trivialization we proceed as follows. On U_j corresponding to a negative odd puncture we first make a positive π-rotation inside Λ begining at

$$(t(\zeta), e^{i\theta(\zeta)})$$

and ending at

$$(-t(\zeta), -e^{i\theta(\zeta)}),$$

and then continue like that to the puncture. On U_1 corresponding to a positive odd puncture we start out at the puncture with the framing

$$(-t(\zeta), -e^{i\theta(\zeta)})$$

make a negative π-rotation inside Λ ending up at

$$(t(\zeta), e^{i\theta(\zeta)}),$$

and then continue like that until we get into the region where the trivialization was already defined.

4.5.5. *Inducing orientations on D.* Recall from Subsection 3.3 the capping disk R_{no}, R_{po}, R_{ne}, R_{pe}. The determinant bundles over these types of boundary conditions are in general *not* orientable. Consider for example the split boundary condition R_{ne} with one-dimensional kernel and zero-dimensional cokernel. The kernel is spanned by a function with second coordinate equal to 0. Now, applying a uniform π-rotation to the first or second line in the split Lagrangian boundary condition brings us back to the original boundary condition. Transporting the orientation along such a path changes it when the first coordinate is rotated and does not change it when the second coordinate is rotated. Thus the bundle is non-orientable. However, the determinant bundles over subspaces of the spaces of capping disks are orientable. Let Y_{no}, Y_{ne}, Y_{po}, and Y_{pe} be the spaces of capping disks such that the second component of the trivialization at the puncture equals $e^{\pm i\theta_0}$.

Lemma 4.37. *The determinant bundles of the $\bar{\partial}$-operator over Y_{no}, Y_{ne}, Y_{po}, and Y_{pe} are orientable.*

Proof. Note that all spaces above are homotopy equivalent to $SO(2)$ and that the monodromy of a generating loop in $\pi_1(SO(2))$ preserves orientation. $\square$

Fix orientations on the bundles over Y_{po} and Y_{pe}. This determines orientations over Y_{no} and Y_{ne}, respectively by requiring that the orientations (of two glueable representatives) glue to the canonical orientation of the determinant of the resulting trivialized boundary condition over the closed disk.

Lemma 4.38. *Diagram orientaions of Λ_u which vary continuously with $u \in X_{m+1}$ induce an orientation on D by gluing capping disks (with orientations as above) to the corners of u.*

Proof. This construction has already been discussed, see Subsection 3.4.2. The result of gluing capping disks R_{no}, R_{ne}, R_{po}, and R_{pe} to the corresponding punctures is a trivialized boundary condition on the closed disk. In the gluing sequence the operators of the capping disk have oriented determinant bundles which together with the canonical orientation on the closed disk induce an orientation on $\bar{\partial}_{\Lambda_u}$. This construction is clearly continuous in u. $\square$

4.5.6. *Basic orientations and sign computation.* To compute signs in contact homology we need to compare the orientation on D induced from the orientation on E and the orientation on D induced by a diagram trivialization. To this end we look at the details of basic orientations to derive the shading rules. We first list the basic orientation choices.

- First we choose an orientation of $\mathbb{R}^2$. This orientation will be fixed throughout the discussion.
- Second we choose an orientation on $\mathbb{C}$.
- Third choose orientations on Y_{po} and Y_{pe}.

Recall that the first two choices determines canonical orientations on the determinant bundles over trivialized boundary conditions over the closed disk. Then the third choices induced orientations on Y_{no} and Y_{ne} respectively via gluing. With all capping operators oriented we orient all the bundles D over X_{m+1}. We next describe how the choices of orientation of $\mathbb{C}$ changes canonical orientations. Recall that the canonical orientation on the determinant bundle over trivialized boundary conditions on the 0-punctured disk was defined by, after deformation, expressing the operator as an operator over D^2 with constant $\mathbb{R}^2$ boundary conditions and an operator over $\mathbb{C}P^1$ with complex kernel or cokernel attached at the origin of D^2. The kernel and cokernel of the latter gets oriented by viewing them as complex vector spaces. Now if the orientation of $\mathbb{C}$ is switched then the orientation of each odd dimensional complex vector space changes and the orientation of each even dimensional complex vector space remains the same. Since the dimension of the kernel and cokernel of the operator over D^2 with constant $\mathbb{R}^2$ boundary conditions is 2 and 0, respectively, the effect on the canonical orientations is as follows: changing the orientation of $\mathbb{C}$ changes the canonical orientation for each operator of index $4j$ and keeps the orientation of each operator of index $4j + 2$. Recall the index of the operators $R_{ne}, R_{no}, R_{pe}, R_{po}$ are $1, 0, -1, 0$, respectively. Fix an orientation of $\mathbb{C}$. Consider $u \in X_1$ with one positive odd corner and with a diagram orientation which agrees with that of the boundary of the disk. Pick the orientation of the determinant bundle over Y_{po} such that when gluing R_{po} to the trivialized boundary condition just described the orientation induced on D agrees with the one induced on D by the bundle E of linearized conformal structures. Note that the operator obtained by gluing R_{po} has index 2 and hence the same choice of orientation works also for the other orientation of $\mathbb{C}$. Denote this orientation o_{po}

Lemma 4.39. *Let $u \in X_1$ be any disk with any diagram orientation. The orientation induced on the fiber of D over u obtained by gluing R_{po} with orientation o_{po} agrees with the orientation induced by E.*

Proof. There are two possible diagram orientations for such a disk. For diagram orientations which can be obtained continuously from the diagram orientation given above the lemma is clear. To see that the lemma holds also for other diagram trivializations we need only check it for one. To this end we rotate the first coordinate of the glued boundary condition by π. This transports the canonical orientation to the canonical orientation and the orientation of the capping disk to the right orientation. Moreover, it preserves the orientation on the 2-dimensional kernel of the $\bar{\partial}$-operator on the one punctured disk. Thus the induced orientations on u with the two different diagram trivializations must be the same. $\square$

Our main theorem (Theorem 4.32) clearly follows from the following lemma.

Lemma 4.40. *Let o_D be the orientation induced on D from the diagram trivialization corresponding to a diagram orientation and let o_E be the orientation induced by E. For a choice of basic orientations $o^A(\mathbb{C})$ on $\mathbb{C}$, o_{po} on Y_{po}, and o_{pe}^A on Y_{pe} we have*

$$o_D = (-1)^{s_A} o_E,$$

where s_A is the number of A-shaded punctures in the diagram orientation. For the choice of basic orientations $o^B(\mathbb{C}) = -o^A(\mathbb{C})$, o_{po}, and $o^B_{pe} = o^A_{pe}$ we have

$$o_D = (-1)^{s_B} o_E,$$

where s_B is the number of B-shaded punctures in the diagram orientation.

Proof. We will prove this lemma by induction on the number of punctures in the disk. As a "base case" we must understand the orientations on one- and two-punctured disks. The lemma for the disk with one positive odd puncture and no other punctures is simply a restatement of Lemma 4.39. Fix an orientation $\hat{o}$ on $\mathbb{C}$. To determine the sign of an arbitrary diagram orientation we will use orientations of 1-dimensional moduli spaces and in particular Lemma 4.11. More precisely, we consider holomorphic disks with several convex and one concave corner. Exactly as above one may use capping disks to orient the one-parameter family of disks in which it lives. For such model families Lemma 4.11 relating various orientations hold. To determine the sign of the disk with one even positive corner and one odd negative corner, consider the disk with one concave (even) corner shown in Figure 2. It is easy to arrange a one-parameter family which splits as follows. One splitting gives a one punctured
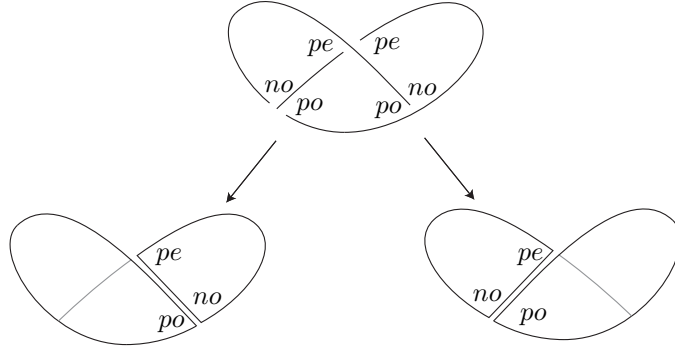

FIGURE 2. Splitting of a one dimensional space of disks.

disk D and a two punctured disk B. The other splitting gives a one punctured disk D and a two punctured disk B'. From the above we know the signs on D and D' are both $+$. Now Lemma 4.11 says we must have

$$\text{sign}(B) = -\text{sign}(B').$$

Note that B and B' are the two possible twice punctured disks with a positive even puncture and a negative odd puncture. Thus we know these disks have opposite signs. One choice of orientation o^A_{pe} yields the A-shading rule and the other one o^B_{pe} the B-shading rule. Note that the operator obtained by capping off B has index 0 and hence its canonical orientation changes with $\hat{o}$. Since the gluing of R_{po} and R_{no} also gives an operator of index 0 also o_{no} changes with the orientation of $\mathbb{C}$ and thus o^A_{pe} and o^B_{pe} are independent of $\hat{o}$.

We have chosen orientations on the determinate line bundles over Y_{pe} and Y_{po} so that the lemma is true for the cases considered so far. To finish the "base case" of our induction we are left to check that the lemma holds for disks B'' and B''' with one positive odd puncture and one negative even puncture. Arguing as in the above paragraph we see that the two configurations of such disks must have opposite signs. Moreover, with respect to one choice of signs on these disks the A-shading rule in the lemma is correct with respect to the other the B-shading rule is. We finish the base case by checking that a change of $\hat{o}$ changes the signs of B'' and B'''. Note that the operator obtained by capping off B'' has index index 2. Thus its canonical orientation is not affected by $\hat{o}$. However, the operator obtained by gluing

R_{pe} and R_{ne} has index 0 so the orientation o_{ne}^A on Y_{ne} induced from o_{pe}^A changes with $\hat{o}$. It follows that there is a choice $\hat{o} = o^A(\mathbb{C})$ so that the A-shading rule holds in all base cases. Moreover, with $o^B(\mathbb{C}) = -o^A(\mathbb{C})$ we find since $o_{pe}^B = -o_{pe}^A$ that the induced orientation o_{ne}^B on Y_{ne} satisfies $o_{ne}^B = o_{ne}^A$ and that the B-shading rule applies in all base cases for $\hat{o} = o^B(\mathbb{C})$. To finish the proof we use the same argument in both cases. For simplicity we give it only in the case of A-shading. Assume by induction that the lemma is true for all disks with n punctures. Let u be a disk with $n + 1$ punctures. If the puncture counterclockwise of the positive puncture is odd then we can glue a once punctured disk D to it. One of the four possible cases if shown in Figure 3. We will finish the argument in this case. The remaining
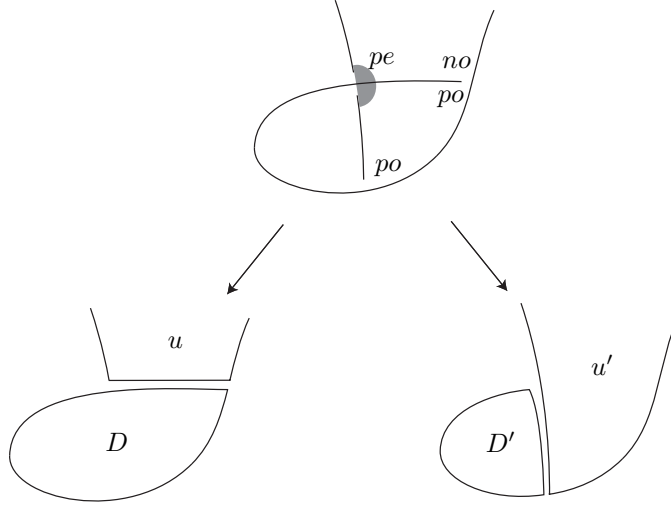

FIGURE 3. Inductive step with a negative corner odd. In the top drawing all corners are shaded.

cases are similar. The one dimensional moduli space formed by this gluing also splits into an n punctured disk u' and a twice punctured disk D'. By induction we know the sign of u' is $(-1)^{s_A'}$, where s_A' is the number of A-shaded regions in u'. Moreover, the sign of D' is $+$. Thus Lemma 4.11 implies

$$\text{sign}(u) = \text{sign}(u)\text{sign}(D) = (-1)\text{sign}(u')\text{sign}(D') = (-1)^{s_A'+1}.$$

Note the number of A-shaded regions of u is $s_A' + 1$. Thus the lemma holds for u. An entirely analogous argument works if the puncture counterclockwise of the positive puncture of u is even. The only difference is one must glue a twice punctured disk to u. (Note in this case one must actually have determined the signs on thrice punctured disks, but this may be done as in the previous paragraph by noting that at least one of the punctures must be odd.)   □

5. LOCAL MORSE THEORY

In this section we describe how to perturb highly degenerate Legendrian submanifolds into generic submanifolds. This will be our main tool in constructing examples in the next section. In Subsection 5.1 we perturb a Legendrian submanifold, whose complex projection has double points along the interior of a compact codimension zero submanifold with boundary and a certain behavior at this boundary using a Morse function on the submanifold, into a generic Legendrian submanifold. For the perturbed Legendrian there will be a double point in the complex projection for each critical point of the Morse function. Adapting a construction of Floer [10] and Pozniak [15], we show that the contact homology boundary map for these

double points is related to the Morse "gradient flow" boundary map. In Subsection 5.2 we compare the signs in these two boundary maps.

5.1. **Perturbing Degenerate Legendrian Submanifolds.**

Consider a Legendrian submanifold L in $\mathbb{R}^{2n+1}$. Let U be an n-manifold with boundary ∂U and let $\phi \colon E \subset J^1(U) \to \mathbb{R}^{2n+1}$ be a contact embedding (also respecting contact forms) such that $\Pi_{\mathbb{C}} \colon U \to \mathbb{C}^n$ is an embedding. Let $\partial U \times [0,2]$ be a collar neighborhood of ∂U in U, with ∂U corresponding to $\partial U \times \{2\}$. Let $V = U \setminus (\partial U \times [1,2])$, $W = U \setminus (\partial U \times [\frac{3}{2},2])$, and $W' = U \setminus \partial(U) \times [\frac{1}{2},2]$. Assume that $L \cap \phi(E)$ consists of two sheets L_1 and L_2 represented in coordinates of $J^1(U)$ as the 1-jet extensions $j^1(f_1)$ and $j^1(f_2)$ of two functions f_1 and f_2, respectively. Assume further that $f_2(x) = f_1(x) + c, c$ a positive constant, for $x \in \overline{V}$, that $df_2 \neq df_1$ for points in $U \setminus \overline{V}$, and that $f_2(p,t) - f_1(p,t)$ is monotone in t for $(p,t) \in \partial U \times (1,2]$. We will frequently think of U and V as subsets in $L_1 \subset L$. We will consider functions $h : U \to \mathbb{R}$ satisfying

(1) h is a Morse-Smale function on W
(2) the support of h contains V and is contained in W
(3) all critical points of $f_2 + h - f_1$ are critical points of h and occur in W'
(4) h is real analytic near its critical points.

Define the Legendrian submanifold L_h to be the one obtained from L by replacing $j^1(f_2)$ in the front projection of L with $j^1 f_2 + h$. The double points of $\Pi_{\mathbb{C}}(L_h)$ which lie inside $\Pi_{\mathbb{C}}(E)$ correspond to the critical points of h and double which lie outside $\Pi_{\mathbb{C}}(E)$ corresponds to double points of $\Pi_{\mathbb{C}}(L \setminus (j^1(f_1)(U) \cup j^1(f_2)(U))$. Denote the part of $\Pi_{\mathbb{C}}(L_h)$ corresponding to $j^1(f_1)(U)$ by U_{f_1} (and similarly for U_{f_2+h}, V_{f_1}, etc.). Having identified the double points of $\Pi_{\mathbb{C}}(L_h)$ we know the generators of its algebra $\mathcal{A}(L_h)$. We next consider holomorphic disks. Let x and y be distinct critical points of h, thought of as double points of $\Pi_{\mathbb{C}}(L_h)$. Let $(s,t) \in \Theta = \mathbb{R} \times [0,1]$ be conformal coordinates for D_2 and N a small tubular neighborhood of U_{f_1} in $\mathbb{C}^n$ that contains U_{f_2+h}. Define

$$\mathcal{N}_h(x,y) = \{u : (\Theta, \partial\Theta) \to N, U_{f_1} \cup U_{f_2+h}| \text{ satisfying (1)–(3)}\}$$

where

(1) $\bar{\partial} u = 0$
(2) $\lim_{s\to-\infty} u(s,t) = x$ and $\lim_{s\to\infty} u(s,t) = y$ and
(3) $u(\mathbb{R} \times \{0\}) \subset U_{f_1}$ and $u(\mathbb{R} \times \{1\}) \subset U_{f_2+h}$.

To define the operator $\bar{\partial}$ we use the standard complex structure on $\mathbb{C}^n$.

Theorem 5.1. *With the notation established above, there exists an $\epsilon > 0$ such that if the C^2 norm of h is less than ϵ, and N is contained in an ϵ neighborhood of U_{f_1}, for positive λ sufficiently small there is a one to one correspondence between $\mathcal{N}_{\lambda h}(x,y)$ and gradient flow lines of λh connecting x to y.*

Our strategy for proving Theorem 5.1 is to transplant our problem into the cotangent bundle of some manifold and then use a slightly modified version of a construction of Floer. To this end we recall Floer's construction [10]. Given a manifold K consider its cotangent bundle $X = T^*K$ thought of as a symplectic manifold. We will denote the zero section of X by K_0. Fix any almost complex structure J on X that agrees with the canonical one along the zero section. Let h be a Morse function on K and define $K_h \subset X$ to be the graph of dh. Let $\pi : X \to K$ be the projection map and set $H(x) = h(\pi(x))$. The Hamiltonian H generates a flow ϕ_t on X and $\phi_1(K_0) = K_h$. Define the time dependent almost complex structure $J_t = \phi_t^* J$ and

$$\mathcal{N}_h'(x,y) = \{u(\Theta, \partial\Theta) \to X| \text{ satisfying (1)–(3)}\}$$

where

(1) $\frac{\partial u(s,t)}{\partial s} + J_t(u(s,t))\frac{\partial u(s,t)}{\partial s} = 0$

(2) $\lim_{s\to-\infty} u(s,t) = x$ and $\lim_{s\to\infty} u(s,t) = y$ and

(3) $u(\mathbb{R} \times \{0\}) \subset K_0$ and $u(\mathbb{R} \times \{1\}) \subset K_h$.

In [10] it was shown that if the C^2 norm of h is sufficiently small then the map $u \mapsto u(s,0)$ from $\mathcal{N}_h'(x,y)$ to $C^\infty(\mathbb{R}, K)$ is a bijection onto the set of bounded trajectories of the gradient flow of h connecting x and y. Moreover if h is a Morse-Smale function then the moduli space $\mathcal{N}_h'(x,y)$ is transversely cut out by its defining equation. Define $\mathcal{N}_h''$ just as $\mathcal{N}_h'$ except instead of the equation in condition (1) use the equation $\bar{\partial}_J u = 0$. The modification of Floer's construction we need is given in the following theorem.

Lemma 5.2. *If the C^2 norm of h is sufficiently small then there is a one to one correspondence between $\mathcal{N}_{\lambda h}'(x,y)$ and $\mathcal{N}_{\lambda h}''(x,y)$ for any λ sufficiently small.*

Proof. Let ϕ_t^λ be the flow of the Hamiltonian $\lambda h(\pi(x))$ and $J_t^\lambda = (\phi_t^\lambda)^* J$. In local Darboux coordinates one may compute that

$$J_t^\lambda = J + \begin{pmatrix} -t\lambda d^2 h & 0 \\ t^2 \lambda^2 (d^2 h)^2 & t\lambda d^2 h \end{pmatrix}$$

where $d^2 h$ is the Hessian of h. Now consider the two parameter family of complex structures

$$J_t^{\lambda,s} = J + \begin{pmatrix} -ts\lambda d^2 h & 0 \\ t^2 s^2 \lambda^2 (d^2 h)^2 & ts\lambda d^2 h \end{pmatrix}.$$

Note $J_t^{\lambda,0} = J$ (independent of λ). So for a fixed λ, $J_t^{\lambda,s}$ interpolates between J_t^λ and J. Define $\mathcal{N}_{\lambda h,s}'(x,y)$ just as $\mathcal{N}_{\lambda h}'(x,y)$ (including boundary conditions) except use the complex structure $J_t^{\lambda,s}$ in condition (1).

Claim. *For λ sufficiently small the spaces $\mathcal{N}_{\lambda h,s}'(x,y)$ are transversely cut out for all $s \in [0,1]$.*

This claim establishes the lemma since it will set up a one to one correspondence between $\mathcal{N}_{\lambda h}'(x,y) = \mathcal{N}_{\lambda h,1}'(x,y)$ and $\mathcal{N}_{\lambda h}''(x,y) = \mathcal{N}_{\lambda h,0}'(x,y)$. To prove the claim note that since h is Morse-Smale we know $\mathcal{N}_h'(x,y)$ is transversely cut out by its defining equation [10]. Said another way, $\bar{\partial}_{J_t}$ is a regular operator. If the claim is not true then we will find a sequence of complex structures converging to J_t that are not regular. But this is a contradiction since by Gromov compactness (and the upper semi-continuity of the dimension of the kernel) we know there is an ϵ' neighborhood of J_t in the space of almost complex structures that contain only regular complex structures. Assume the claim is false. So for all $\lambda > 0$ there is some s so that $\mathcal{N}_{\lambda h,s}'(x,y)$ is not transversely cut out. Thus there is always a solution $\widetilde{u} \in \mathcal{N}_{\lambda h,s}'(x,y)$ for which $D_{\widetilde{u}}\bar{\partial}_{J_t^{\lambda,s}}$ is not surjective. Since $\widetilde{u}$ satisfies

$$\bar{\partial}_{J_t^{\lambda,s}}\widetilde{u} = d\widetilde{u} + J_t^{\lambda,s} \circ d\widetilde{u} \circ j = 0$$

the map $u = \phi_t^{1-\lambda}(\widetilde{u})$ has boundary conditions K_0 and K_h and satisfies

$$du + \widetilde{J}_t^{\lambda,s} \circ du \circ j = 0.$$

In addition, since $\phi_t^{1-\lambda}$ intertwines the complex structures $J_t^{\lambda,s}$ and $\widetilde{J}_t^{\lambda,s}$ we see that $\widetilde{J}_t^{\lambda,s}$ is not regular. One may compute that $\widetilde{J}_t^{\lambda,s} - J_t$ is of order λ so for λ small enough $\widetilde{J}_t^{\lambda,s}$ must be regular. Thus for λ small enough $J_t^{\lambda,s}$ is regular for all $s \in [0,1]$. $\square$

To finish the proof of Theorem 5.1 consider the manifold $K = \overline{U} \cup_{\partial\overline{U}} -\overline{U}$ and its cotangent space $X = T^*K$. We can find a symplectomorphism ψ from a neighborhood N of U_{f_1} in $\mathbb{C}^n$ to a neighborhood N' of $U \subset X$, thought of as part of the zero section, that takes U_{f_1} to U and sends the standard complex structure along $U_{f_1} \subset \mathbb{C}^n$ to the canonical complex

structure along the zero section. (To ensure that U_{f_2+h} still sits in N one merely needs to make sure the C^1 norm of h is sufficiently small.) Now let J be the complex structure on X that extends the one pushed forward from N and is standard along the entire zero section. There is a function $g : K \to \mathbb{R}$ that is constant on $V \subset K$ and satisfies $\Gamma_g \cap T^*U = \psi(U_{f_2})$. We can choose U to be a small enough neighborhood of $\overline{V}$ so that the function g has C^2 norm small enough for Floer's results to hold. We can further find a function $\widetilde{h} : K \to \mathbb{R}$ such that $\Gamma_{g+\widetilde{h}} \cap T^*U = \psi(U_{f_2+h})$. In addition there is some $\epsilon > 0$ so that if $\|h\|_{C^2} < \epsilon$ then $g + \widetilde{h}$ will also have C^2 norm small enough for Floer's results to hold. According to Lemma 5.2 we can now find a λ so that $\mathcal{N}'_{\lambda(g+\widetilde{h})}(x, y)$ is in one to one correspondence with $\mathcal{N}''_{\lambda(g+\widetilde{h})}(x, y)$. It is easy to see that all the holomorphic curves in $\mathcal{N}''_{\lambda(g+\widetilde{h})}(x, y)$ lie in N' and correspond, via ψ, to holomorphic curves in $\mathcal{N}_h(x, y)$. This establishes Theorem 5.1.

5.2. **Comparison with the Morse-Witten complex.** We briefly recall how to compute the homology of a manifold via a Morse function. Given a Morse-Smale function $f : M \to \mathbb{R}$ we get a chain complex generated by the critical points of f. The boundary map of this chain complex comes from counting isolated gradient flow lines between critical points. We assign a sign to each such flow line as follows. Let x and y be two critical points. The Morse index of x being one larger than the Morse index of y. Let U_x be the unstable manifolds of x (which is the set of points in M that under the gradient flow, flow to x as time goes to $-\infty$). and S_y be the stable manifold of y (which is the set of points that flow to y as time goes to ∞). The gradient flow lines connecting x to y are exactly $U_x \cap S_y$. If we chose orientations on U_x for all critical points x and assume M is oriented then we get induced orientations on S_x for all critical points x. If such a choice has been made then $U_x \cap S_y$ is an oriented manifolds. If it is a one dimensional manifold then it also gets an orientation by $-\nabla f$. Thus for isolated flow lines connecting x and y we get a sign by comparing these orientations. The boundary map in the chain complex comes from the signed count of these flow lines. The homology of this complex agrees with the homology of the manifold. For more details see [16]. Consider the set up in Section 5: a Legendrian with an open set V of double points in its Lagrangian projection and a Morse-Smale function h on V with which we perturb the Legendrian. We now wish to compare the signed count of flow lines above with the signs of holomorphic disks associated to the flow lines by Theorem 5.1. To this end choose a tree of gradient flow lines of h that connect all the critical points of h (we are assuming V is connected so this can be done). Each of the moduli spaces of flow lines can be assigned an orientation using the coherent orientations from Section 3.4, by identifying it with the space of holomorphic curves using Theorem 5.1. We may now choose orientations on the unstable manifolds of the critical points so that the orientations induced by intersecting the stable and unstable manifolds agrees with the coherent orientations on the flow lines in the chosen tree. This is possible since we chose a tree of flow lines. Note that the orientations on all other moduli spaces of flow lines are determined by the orientations on the spaces of flow lines in the chosen tree and gluing constructions. Since the orientations on the moduli space of flow lines coming from Section 3.4 and the procedure outlined in the previous paragraph are both "coherent" they are both determined by the orientations on the flow lines in the tree and gluing constructions. We have proved

Theorem 5.3. *Under the correspondence between holomorphic disks and gradient flow lines set up in Theorem 5.1 there is a choice on orientation on the unstable manifolds of h so that the Morse-Smale orientations and orientations from Section 3.4 agree on all moduli space of flow lines.*

5.3. **Examples.** In this section we give examples showing that oriented contact homology is a finer invariant than contact homology over $\mathbb{Z}_2$. To keep computations simple, we give examples

showing that the linearized contact homology over $\mathbb{Z}$ will distinguish Legendrian links in the 1-jet space of T^n not distinguished by the full contact over $\mathbb{Z}_2$. Similar constructions can be applied to construct Legendrian submanifolds in $\mathbb{R}^{2n+1}$; however, the computation of the full contact homology is more difficult. (It relies on a Morse theoretic description of all holomorphic disks relevant to contact homology, involving gradient flow trees with cusps.) We therefore defer discussions of this to a forthcoming paper.

Recall $J^1(T^n) = T^*T^n \oplus \mathbb{R}$ has a natural contact structure $\alpha = \lambda + dt$ where λ is the Liouville 1-from on $T^*(T^n)$ pulled back to $J^1(T^n)$ and t is the coordinate in the $\mathbb{R}$ factor. Projecting out the $\mathbb{R}$ factor is called the Lagrangian projection and is analogous to projecting out the z-coordinate in $\mathbb{R}^{2n+1}$. Note that since $T^*(T^n)$ admits a complex structure coming from $\mathbb{C}^n$ by quotienting by a holomorphic action the analytic set up of contact homology in [5] for $\mathbb{R}^{2n+1} = J^1(\mathbb{R}^n)$ also works for $J^1(T^n)$. Moreover, the proofs in [5] carry over word-for-work to this setting. (Note, in general, setting up contact homology in jet spaces requires more work, see [4].) With the above understood we can take the contact homology of Legendrian links in $J^1(T^n)$ to be well-defined. Our main theorem of this subsection is

Theorem 5.4. *For any $n \geq 4$ there are infinitely many Legendrian links, topologically isotopic to two copies of the zero section, in $J^1(T^n)$ that are distinguished by their oriented linearized contact homology, but have the same classical invariants and tame isomorphic contact homology DGA's over $\mathbb{Z}_2$.*

Let L_0 be the zero section of $J^1(T^n)$ and f a non-negative self-indexing Morse function on T^n with minimal number of critical points. We will now alter f near the unique index 0 critical point. For p odd Figure 4 gives a handle decomposition of D^4. There are clearly
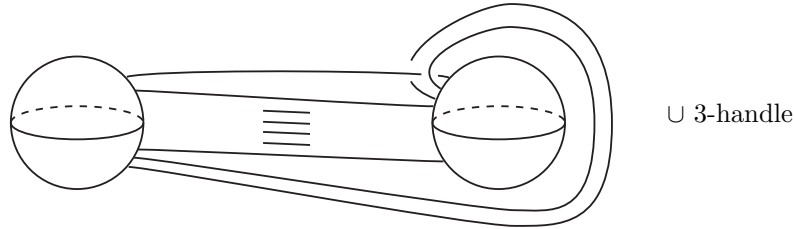

FIGURE 4. Handle decomposition of D^4. One of the 2-handles goes p times over the 1-handle the other goes once.

analogous decompositions for all $n \geq 4$. ¿From this we get a Morse function h on D^n with critical points c_0, c_1, c_2, c_2', c_3. The critical points are labeled by their Morse index. We can assume that $h(c_0) < h(c_1) < h(c_2) < q < h(c_2') < h(c_3)$ and $h^{-1}(\frac{1}{2}) = \partial D^4$. Moreover, the boundary map for the Morse-Witten complex is

$$\partial c_0 = 0 \qquad \partial c_2' = c_1$$
$$\partial c_1 = 0 \qquad \partial c_3 = pc_2' - c_2$$
$$\partial c_2 = pc_1$$

Let f_p be the Morse function on T^n equal to f on $f^{-1}((-\infty, \frac{1}{2}])$, that is, outside a neighborhood of the index 0 critical point of f, and in the neighborhood equal to h. We can assume f_p is smooth. Let L_p be the 1-jet of $\epsilon(f_p - q)$ in $J^1(T^n)$:

$$L_p = \{(x, \epsilon df_p(x), \epsilon(f_p(x) - q))\}.$$

We now fix the index 0 critical point and measure all gradings relative to this. According to Theorem 5.1 and the discussion following it we know, for ϵ small enough, all the holomorphic

disks coming in to the computation of the boundary map come form gradient flow lines. Thus one may easily compute the boundary map of $L_0 \cup L_p$ on the generators associated to h are

$$\partial c_0 = 0 \qquad\qquad \partial c_2' = 0$$
$$\partial c_1 = pc_2 \qquad\qquad \partial c_3 = pc_2'$$
$$\partial c_2 = 0$$

Moreover, the boundary map on the other generators comes from the Morse boundary map for f. Thus when everything is reduced mod 2 the boundary map is independent of p but the contact homology over $\mathbb{Z}$ (with relative grading chosen so that c_0 has grading n) of the link $L_0 \cup L_p$, when $n \geq 8$, is

$$LCH_i(L(p)) = \begin{cases} \oplus_{n+1}\mathbb{Z}, & i = 1 \\ \oplus_{(n+1)n}\mathbb{Z} \oplus \mathbb{Z}_p & i = 2 \\ \oplus_{(n+1)n(n-1)}\mathbb{Z} & i = 3 \\ \oplus_{(n+1)n(n-1)(n-2)+1}\mathbb{Z} & i = 4 \\ \oplus_{\binom{n+1}{i+1}}\mathbb{Z} & 4 < i < n-3 \\ \oplus_{(n+1)n(n-1)+1}\mathbb{Z} & i = n-3 \\ \oplus_{(n+1)n}\mathbb{Z} \oplus \mathbb{Z}_p & i = n-2 \\ \oplus_{n+1}\mathbb{Z} & i = n-1 \\ \oplus_2\mathbb{Z} & i = n. \end{cases}$$

One obtains a similar answer when $4 \leq n < 8$. Note these gradings are only relative gradings, but this is irrelevant as the p torsion distinguishes the Legendrian links independent of grading. The gradings were computed using the Morse index, see [5].

6. Double points of exact Lagrangian immersions

6.1. **Double point estimates for Legendrian submanifolds with good algebras.** We describe how to use contact homology to derive a lower bound on the number of double points of an exact Lagrangian immersion. Contact homology has also been used to study intersections of a pair of immersions [1]. Let $f \colon L \to \mathbb{C}^n$ be an exact Lagrangian immersion. Then after small perturbation we may assume that the Legendrian lift of L is an embedding which is chord generic. Let $(\mathcal{A}, \partial)$ denote its DGA. Recall a DGA is augmented if the differential of no generator contains a constant. An *augmentation* of an algebra is a graded map $\epsilon \colon \mathcal{A} \to \mathbb{Z}$ such that $\epsilon(1) = 1$ and $\epsilon \circ \partial = 0$. Given an augmentation ϵ the graded algebra tame isomorphism $\phi_\epsilon(a) = a + \epsilon(a)$ will conjugate $(\mathcal{A}, \partial)$ to an augmented algebra. A DGA is called *good* if it admits an augmentation, and is hence tame isomorphic to an augmented DGA. We show the following theorem where we use the algebra $\mathcal{A}(L)$ with coefficients other than $\mathbb{Z}$, see Remark 4.2.

Theorem 6.1. *Let $f \colon L \to \mathbb{C}^n$ be an exact Lagrangian immersions and let $(\mathcal{A}, \partial)$ be the DGA associated to an embedded chord generic Legendrian lift of f. If $(\mathcal{A}, \partial)$ is good then f has at least*

$$\frac{1}{2}\dim(H_*(L; \Lambda))$$

double points, where $\Lambda = \mathbb{Q}$ or $\Lambda = \mathbb{Z}_p$ for any prime p if L is spin and $\Lambda = \mathbb{Z}_2$ if L is not spin.

Proof. To simplify notation we identify L with its image under the embedding which is the lift of f and write $L \subset \mathbb{C}^n \times \mathbb{R}$. Let L' be a copy of L shifted a large distance in the z-direction, where as usual z is a coordinate in the $\mathbb{R}$-factor. Then $L \cup L'$ is a Legendrian link.

Moreover, assuming that the shifting distance in the z-direction is sufficiently large, shifting L' s units in the x_1-direction gives a Legendrian isotopy of $L \cup L'_s$. After a large such shift $L \cup L'_s$ projects to two distant copies of $\Pi_{\mathbb{C}}(L)$ and it is evident that an augmentation for L gives an augmentation for $L \cup L'_s$. Moreover, the linearized contact homology of $L \cup L'$ equals the set of sums of two vector spaces from the linearized contact homology of L. We will next compute the linearized contact homology of $L \cup L'$ in a different manner. Let $g \colon L \to \mathbb{R}$ be a Morse function on L and use g to perturb L in $U \subset J^1(L)$, where U is a small neighborhood of the 0-section. After identification of U with a neighborhood of L' in $\mathbb{R} \times \mathbb{C}^n$ we use this isotopy to move L' to L''. The projection of L'' into $\mathbb{C}^n$ then agrees (locally) with an exact deformation of L in its cotangent bundle and there is a symplectic map from that cotangent bundle to a neighborhood of $\Pi_{\mathbb{C}}(L)$ in $\mathbb{C}^n$. Pulling back the complex structure from $\mathbb{C}^n$ we get an almost complex structure on T^*L. The intersection points of L and L'' are of three types.

(1) Critical points of g.
(2) Pairs of intersection points between L and L'' near the self-intersections of L.
(3) Self intersection points of L and of L'' near self intersections of L.

Fix augmentations of $\mathcal{A}(L)$ and of $\mathcal{A}(L'')$. If ∂ is the differential of $\mathcal{A}(L \cup L'')$ it is easy to see that any monomial in ∂c, where c is a Reeb chord of type (1) or (2) must contain an odd number of Reeb chords of type (1) and (2). Therefore the augmentations of $\mathcal{A}(L)$ and $\mathcal{A}(L'')$ give an augmentation for $\mathcal{A}(L \cup L'')$ that is trivial on double points of type (1) and (2). Denote by d the linearized differential induced by the augmentations chosen and by E_i the span of the double points of type (i), $i = 1, 2, 3$. Suppose a is a type (3) double point, then ∂a has no constant part and its linear part has no double points of type (1) or (2), since each holomorphic disk with a positive puncture at a must have an even number of negative corners of type (1) or (2). Thus $d(E_3) \subset E_3$. If b is of type (1) or (2) then the linear part of ∂b involves only double points of type (1) and (2). Denote by π_i the projection onto $E_i, i = 1, 2$, and $d_i = \pi_i \circ d$. Then $d = d_1 + d_2$ on $E_1 \oplus E_2$. Consider $d_1 \colon E_1 \to E_1$. We claim that for sufficiently small perturbation g, $d_1 \circ d_1 | E_1 = 0$. To show this we consider gluing of two (two-punctured) disks contributing to d_1. This gives a 1-parameter family two-punctured disks. Now, for sufficiently small perturbation, no Reeb chord of type (2) has length lying between the lengths of two Reeb chords of type (1). Moreover, every Reeb-chord of type (3) has length bigger than the difference of the lengths of two Reeb chords of type (1). This shows that the 1-parameter family must end at another pair of broken disks with corners of type (1). It follows that $d_1^2 = 0$. It follows that $d_1 | E_1$ agrees with the Floer differential of $\hat{L} \cup \hat{L}_g$, where $\hat{L} \subset T^*L$ is the 0-section and where $\hat{L}_g \subset T^*L$ is the graph of dg. Hence,

$$\mathrm{Ker}(d_1 | E_1)/\mathrm{Im}(d_1 | E_1) \approx H_*(L; \Lambda).$$

Write $E_1 = W \oplus V$, where $W = \mathrm{Ker}\, d_1 | E_1$ and let W' be a direct complement of $d_1(V) \subset W$. Then $\dim W' = \dim H_*(L; \Lambda)$. Fix the augmentations for L and L' which gives the element of the linearized contact homology of L which has the largest dimension. By the above discussion we find that $\mathrm{Ker}(d_3)/\mathrm{Im}(d_3)$ equals a direct sum of two copies of this maximal dimension vector space. It follows that the contribution to the linearized contact homology involving double points between L'' and L must vanish. We check how double points of type (2) kill off the double points of type (1) that exist in the homology of (E_1, d_1). We compute

$$0 = d(d(W')) = \pi_1(d(d(W')))$$
$$= \pi_1(d(d_2(W'))) = d_1(d_2(W')),$$

where the third equality is due to the fact that $W' \subset E_1$ is in $\mathrm{Ker}(d_1 | E_1)$. It follows that $\mathrm{Im}(d_2 | W') \subset \mathrm{Ker}(d_1 | E_2)$. Moreover, notice an element e in W' is a non zero element in the

linearized contact homology if and only if $d_2e = 0$ and $e \notin \mathrm{Im}(d_1|E_2)$. Thus if $d_2e = 0$ then e is in $\mathrm{Im}(d_1|E_2)$, showing that $\mathrm{Ker}\, d_2|W' \subset \mathrm{Im}\, d_1|E_2$. We find

$$\dim(E_2) = \dim(\mathrm{Ker}\, d_1|E_2) + \dim(\mathrm{Im}\, d_1|E_2) \geq$$
$$\dim(\mathrm{Im}\, d_2|W') + \dim(\mathrm{Ker}\, d_2|W') = \dim(W'),$$

and conclude that

$$2 \cdot \sharp\{\text{double points}\} = \dim(E_2) \geq \dim(W') = \dim(H_*(L; \Lambda)).$$

$\square$

6.2. Improving double point estimates.
In this section we show how to remove a constant from a double point estimate.

Theorem 6.2. *Suppose there is a constant K such that for any exact Lagrangian immersion $f\colon L \to \mathbb{C}^n$, f has at least*

$$\frac{1}{2}\dim(H_*(L; \Lambda)) - K$$

double points, where $\Lambda = \mathbb{Q}$ or $\Lambda = \mathbb{Z}_p$ for any prime p if L is spin and $\Lambda = \mathbb{Z}_2$ if L is not spin. Then f has at least

$$\frac{1}{2}\dim(H_*(L; \Lambda))$$

double points.

We first prove a simple lemma that is a generalization of a "spinning" operation in [5].

Lemma 6.3. *Let $f\colon L \to \mathbb{C}^n \times \mathbb{R}$ be a chord generic Legendrian embedding with $R(f)$ Reeb chords and Maslov number m_f. Then, for any $k \geq 1$ there exists a Legendrian embedding $F_k\colon L \times S^k \to \mathbb{C}^{n+k} \times \mathbb{R}$ with $2R(f)$ Reeb chords and $m_{F_k} = m_f$.*

We call the Legendrian embedding of $L \times S^k$ the k-spin of L.

Proof. For $q \in L$, let $f(q) = (x(q), y(q), z(q))$. Note that translations in the x_j-direction, $j = 1, \ldots, n$ and that the scalings $x \mapsto kx$, $z \mapsto kz$, $k \geq 0$ are Legendrian isotopies which preserves the number of Reeb chords. We may thus assume that $f(L)$ is contained in $\{(x, y, z)\colon |x| \leq \epsilon\}$, where ϵ is very small. For convenience, we write $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ with coordinates $x = (x_0, x_1)$ and corresponding coordinates (x_0, y_0, x_1, y_1) in $T^*\mathbb{R}^n = \mathbb{C}^n$. Consider the embedding $S^k \subset \mathbb{R}^{k+1} \subset \mathbb{R}^{k+n}$, where S^k is the unit sphere in $\mathbb{R}^{k+1}$. Let $(\sigma, x_0, x_1) \in S^k \times \mathbb{R}_+ \times \mathbb{R}^{n-1}$

$$(\sigma, x_0, x_1) \mapsto x_0 \cdot \sigma + x_1,$$

be polar coordinates on $\mathbb{R}^{k+n}$. Fix a Morse function ϕ, with one maximum and one minimum on S^k which is an approximation of the constant function with value 1. Define $F\colon S^k \times L \to \mathbb{C}^{n+k} \times \mathbb{R}$, $F = (F_x, F_y, F_z)$ as follows

$$F_x(\sigma, q) = (1 + x_0(q)) \cdot \sigma + x_1(q),$$
$$F_y(\sigma, q) = (1 + x_0(q))^{-1}\nabla_{S^k}\phi(\sigma) + \phi(\sigma)(y_0(q) \cdot \sigma + y_1(q)),$$
$$F_z(\sigma, q) = \phi(\sigma)z(q),$$

where we think of the gradient $\nabla_{S^k}\phi(\sigma)$ as a vector in $\mathbb{R}^{k+1}$ tangent to S^k at σ. It is then easily verified that F is a Legendrian embedding. Moreover, the Reeb chords of F occur between points (q, σ) and (q', σ') such that $\sigma = \sigma'$, $x_j(q) = x_j(q')$, $y_j(q) = y_j(q')$, $j = 0, 1$, and either $z(q) = z(q')$ or $\nabla_{S^k}\phi(\sigma) = 0$. However, these conditions are incompatible with f being an embedding unless $\nabla_{S^k}\phi(\sigma) = 0$ and we conclude that the number of double points of F are as claimed. The statement about the Maslov number is straightforward. $\square$

Proof of Theorem 6.2. Given K in the statement of the theorem, choose l so that $2^l > K$. For any immersed exact Lagrangian $f \colon L \to \mathbb{C}^n$ lift f to an embedded Legendrian in $\mathbb{C}^n \times \mathbb{R}$ and k-spin this Legendrian l times. The Lagrangian projection of the resulting Legendrian gives a new exact Lagrangian immersion $F_k \colon L \times S^k \times \ldots \times S^k \to \mathbb{C}^{n+lk}$. Since Reeb chords correspond to double points F_k has 2^l time as many double points as f. We have

$$2^l R(f) = R(F_k) \geq \frac{1}{2} \dim(H_*(L \times S^k \times \ldots \times S^k; \Lambda)) - K$$

$$= \frac{1}{2}(2^l \dim(H_*(L; \Lambda))) - K.$$

Thus

$$R(f) \geq \frac{1}{2} \dim(H_*(L; \Lambda)).$$

$\square$

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